ASPECTS OF GRAND UNIFIED AND STRING PHENOMENOLOGY

A Dissertation

by

JOEL W. WALKER

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2005

Major Subject: Physics

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Approved by:

Chair of Committee,	Dimitri Nanopoulos
Committee Members,	Richard Arnowitt
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ABSTRACT

Aspects of Grand Unified and String Phenomenology. (August 2005)

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Explored in this report is the essential interconnectedness of Grand Unified and String Theoretic Phenomenology. In order to extract a modeled connection to low-energy physics from the context of superstring theory, it is presently necessary to input some preferred region of parameter space in which to search. This need may be well filled by a parallel study of Grand Unification, which is by contrast in immediate proximity to a wealth of experimental data. The favored GUT so isolated may then reasonably transfer this phenomenological correlation to a string embedding, receiving back by way of trade a greater sense of primary motivation, and potentially enhanced predictability for parameters taken as input in a particle physics context.

The Flipped $SU(5)$ GUT will be our preferred framework in which to operate and first receives an extended study in a non-string derived setting. Of particularly timely interest are predictions for super-particle mass ranges and the interrelated question of proton decay lifetime. Corrections to such a picture under the lift to a string embedding are also considered. Two principal approaches to string model building are next treated in turn: the Heterotic Free Fermionic construction and Intersecting D-branes in Orientifold compactifications. In both contexts, a summary of existing constructions, extensions to known procedures, and original phenomenological contributions are described.

To my parents, my wife Laura and our son Isaac

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INTRODUCTION

A. A View of Science

The underlying goal which most interests me scientifically is a unified understanding of physical law. My favorite story by way of example is the long slow unraveling of electromagnetic theory. From just simple experiments involving furs or chips of amber, the electrostatic interaction might be seen, and reasoned to exploit two distinct states of charge. Elementary graphical methods could suffice to make the next great step, a mathematical abstraction of these observations as an inverse-square force. Likewise, the deflections of compass needles by currents and the induction of currents by the motion of magnets were also measured and codified. It was only from such concise expressions that the master-stroke of Maxwell could emerge: the existing formulae were alone theoretically inconsistent, requiring also a description of induced magnetic fields if electric charge was to be conserved. This insight was followed in short order by laboratory confirmation of electromagnetic waves, traveling in like manner unto light. Only then in the completeness of these equations could their greater structure could be glimpsed, holding already a full representation of the Relativity theory, and properly suited to couple as the locally gauge invariant intermediary of a quantum field theory. The whimsies of experience wherein coiled flows of charge produced a strong magnet, and particles of charged matter traced helical paths through regions of magnetic field, could not then be picked and chosen in isolation. They were instead the unavoidable and integrated consequence of these two well motivated principles. Physical law, it could be imagined, was simply as it must be.

The journal model is *IEEE Transactions on Automatic Control*.

B. The Task at Hand

The central task of physics remains as it has always been: the distillation of information down into knowledge, the union of facts seemingly disparate, by common woven threads of principle. The revelation of modern physics has been that the proper eye with which to perceive this tapestry is that of the ultra-high energy micro-world. With this insight comes also a higher calling to consider not only goodness of law, but further origins of law, and the more basic constructions within which the abstracted content of our effective theories may nest. Smaller yet in comparison to the nuclear scale than the atomic nucleus is smaller than us, mathematics must become the new microscope for this unseen cosmos. Symmetry, unity and simplicity are our lampposts in the search for a framework from which naturally descends the distilled elements of low-energy physics, namely covariant locally gauged chiral matter multiplets in replicated families. They are the lessons learned from study of natural law, the beacons to which seekers of truth are drawn and the search beams beneath which successes have been found.

With simple statements of symmetry the currencies which nature holds dear are imbued into mathematical expression with such power of constraint that little else could hope to be written. Extending the framework to operate locally raises up fully specified interactions from the global charges, and corresponding gauge fields for their mediation. With one stroke the rates of exchange are sealed, and the conceptions of field source and reaction to field are unified in a renormalizable manner. Even the matter particle spectrum is confined to fit neatly within representations of the group structure specified. It is no surprise then that a master group to contain the standard model should be sought in form of a Grand Unified Theory (GUT) which would attempt to merge the known electromagnetic and two nuclear forces into a single conceptual framework. Indeed, the notion that the three gauged symmetries of particle physics should manifest a united structure during extremely energetic interactions is well motivated by precision low-energy measurements.

Under a natural logarithmic renormalization, the tendency toward asymptotic freedom of the non-Abelian $SU(3)_C \times SU(2)_L$ elements steadily drives their gauge couplings downward [1], in cautious balance against the simultaneous ascendance of the hypercharge $U(1)_Y$. Perhaps the most fascinating feature of this program however is the indication that fundamental processes occur some twelve orders of magnitude in energy beyond the scope of our most sophisticated accelerators. This speaks again to complexity arising out of simplicity at a scale alluringly close to the Planck regime where gravitational effects are expected to be strong, with a prevailing quantum structure. The satisfaction of low-energy constraints [2, 3, 4] while using only MSSM field content [2] reveals the vast desert for what it is [5], a great and empty field for the couplings to run in and produce in their crossing our familiar world. However, spanning such unimaginable leagues requires a fundamental shift of paradigm, and tools born to the world in ways unfamiliar.

String theory, with its relatives M-theory and local supersymmetry, is the principal modern candidate for such investigations. Great promise exists in the natural, consistent and necessary accommodation of gravity as well as the tremendous gauge freedom which may be transferred out of the intrinsically resident extra spatial dimensions. Indeed, the defining insight delivered from this paradigm appears to be the commonality of origin between gauge and space-time symmetries. Great shortcoming exists in the same places however, as the theory seems in fact overly malleable, retain-

[1]The weak nuclear coupling α_2 is comparatively sedate in its motion, and in a supersymmetric context actually increases marginally with energy transfer.

[2]Standard Model here also refers to its extensions that include neutrino masses.

ing too great a freedom of parametrization to be truly predictive. Despite the wealth of persistent and generically stable desirable features, any mechanism for prediction of the theorys vacuum remains elusive.

That the string might one day speak its secrets in a language singular and inevitable is indeed a noble dream; unfortunately, a dream as yet well beyond our humble means. While non-perturbative effects must finally produce a dynamic and definite solution [6], it seems there is also a value to searching in the available light of a favored theoretical framework and constraint for the simple existence of a viable, albeit ad-hoc, solution. We must then be content to reveal and study those appealing corners of the theory which are under no prohibition. However, even guided by such strong constraints as preserving $\mathcal{N} = 1$ SUSY and the consistent exclusion of anomalies, there remains too great a freedom for any meaningful comprehensive search. Instead some target of opportunity must be established, and in this regard the existence of a preferred GUT model can be of invaluable assistance.

By contrast to string theory, the GUT pictures are not a system of dynamics in themselves, but contain nonetheless a wealth of new and observable predictions. That most discerning indicator, which has been the death already of many proposals, is that new baryon-number violating processes not over-speed proton decay, while also maintaining consistency with the measured range of $\alpha_s(M_Z)$. Including a light spectrum of superpartners, as independently required for stabilization of the minimally extended Standard Model (MSSM) [7, 8, 9] with a low-mass Higgs [10] and favored cosmologically for cold dark matter [11, 12], improves the first order gauge coupling convergence to the per-mille level [2, 13, 3, 14] while also lavishly providing for phenomenological constraint to threshold corrections in the next order. Surely predictivity is not everything though, and especially in this new climate, insight to context and origin for knowledge previously held is a great achievement in itself. The entanglements of scale, and symbioses of deficiencies suggest then a mutual partnership between these studies. Strings endow particle physics with dynamics and potentially added calculability, while GUTs offer in return a grounding in reality, and a reasoned starting point for model building.

The treatment in this thesis will thus be multi-fold. Firstly, to read seriously the cumulative low energy measurements which can help lock into place the unknown elements of a potential GUT embedding, and secondly to consider the further embedding of the preferred model as a string construction. The discussion will consist of three independent yet essentially interconnected primary topics as introduced following. It will entail some analysis of what GUT constructions the string appears to prefer and also discussion of string corrections to both decay rates and allowed modes. The context of this investigation will be two distinct model building scenarios which have each met prior with some reasonable degree of success: Free Fermions on the Heterotic string [15, 16, 17] and the intersecting D-Brane constructions on orientifold compactifications [18]. There will be a strong focus on extending the available technology within certain corners of each paradigm, and also detailed demonstration of model building procedures. It is hoped the techniques so developed will have enduring relevance, and that the examples unearthed may illuminate some properties of the true path, serving to narrow the gulf from the low-energy side, and set a table in waiting for the weary travelers who cross the desert.

C. Chapter Introductions

1. Grand Unification and Proton Decay

This section describes research in collaboration with Dr. John Ellis and Dr. Dimitri Nanopoulos, updating and extending the study [19].

Beyond this effortless conspiracy of coupling unification, the GUT paradigm is also appealing from a purely theoretical stand-point, naturally extending the known electro-weak mixing, providing an origin for charge quantization, and potentially eliminating some free parameters from the Standard Model. Moreover, the intermingling of Leptonic and Baryonic quantum numbers which presages instability of the proton is, along with CP violation and an of out-of-equilibrium cosmological phase, one of the three ingredients considered essential to a natural providence of matter/anti-matter asymmetry.

Following the healthy philosophy which warns against plurality without necessity, it is no surprise that early searches for a satisfactory Grand Unified Theory landed squarely on the doorstep of $SU(5)$, the most elemental group structure capable of enclosing the requisite rank-4 SM system. Nevertheless, we take no interest here in $SU(5)$, preferring instead the enlarged variation $SU(5)_L \times U(1)_X$. Interestingly, group theory provides only this single alternative embedding into $SU(5)$ while respecting the existing $SU(3)_c \times SU(2)_L \times U(1)_Y$ structure. We are thus instructed of the possibility that the pair of right-conjugate color triplets ($u^c_L \Leftrightarrow d^c_L$) and the pair of singlets ($e^c_L \Leftrightarrow \nu^c_L$) must each flip their locations. Under this redistribution, the true hypercharge is now an admixture of both $U(1)_x$ and the $U(1)$ factor of $SU(5)$.

It is a fair question why after seeking out the minimal Grand Unified gauge group, one would then follow up by attaching this uncomely and decidedly non-minimal appendage. But, in fact, it seems the day of complication has arrived, with standard $SU(5)$ proving insufficient to its testing. The chinks in the armor had been apparent already in the machinations required to cleanly split the triplets from doublets and in the absence of a right-handed neutrino. Once heralded as a great boon, this omission now stands in opposition to the indications of Super Kamiokande from which the study of neutrino oscillations is enough to remind that for nature, beauty is not subject to the eye of the beholder. Since chirality is not Lorentz invariant for massive fields, there must exist a sixteenth state. Shortfalls exist also in the excessively high prediction for the strong coupling at M_z, and now most severely and most fatally in view of recent experiments, in the insufficient lifetime offered to the proton. From each of these fronts, the experimental consensus is resounding: Supersymmetric flipped $SU(5)$ survives where its forbear cannot. In presence of these facts we are reminded of nature's prior dictum on the virtues of delayed gratification at the electroweak scale. Indeed, if vector-like $SU(2)$ had prevailed against chiral $SU(2)_L \times U(1)$ at that juncture, there would be no survival of the low energy physics which pervades our experience against renormalization to the next cutoff. Likewise, it may also be precocious to insist on a strict unification which would preempt the presence of gravitation, especially in light of the nearby string scale from which is generically expected to emerge a unified structure of possibly much larger size than just the MSSM.

Also along for the ride in flipped $SU(5)$ are the procurement of various other phenomenological benefits and observable signatures. Not only is the disastrously rapid $K^+\bar{\nu}$ dimension five proton decay which plagues standard $SUSY$ $SU(5)$ suppressed, but it is replaced as principal by a still viable yet imminently observable proton decay mode in the dimension-6 $e^+\pi$ channel. Furthermore, not only are right-handed neutrinos essentially accommodated, but and eV-scale neutrino mass is also naturally provided by means of the 'see-saw' mechanism.

Still, it should be noticed that all is not lost for the promised benefits of grand unification. We are free to continue the game upward, and ask what group of rank five might contain this favored structure. The answer again is clear, and accompanied by fresh insight and reinforcement for this approach. The group $SO(10)$ can embed all sixteen desired states as a spinor representation. Furthermore, the bar5 so achieved are a perfect match to the variation employed as fermionic matter in flipped $SU(5)$. By contrast, the pair of $\bar{\mathbf{5}}$ split from a fundamental of $SO(10)$ carry the quantum numbers of standard $SU(5)$, exactly as is still needed to serve for Higgs fields. Memory of this true GUT embedding will suffice to preserve in flipped $SU(5)$ all essential related properties. We shall see whether all good things indeed come to those who wait.

Among the recent experimental inputs of relevance to this study are the upwardly shifted top quark mass measurements from DØ, and the precision determinations of the cold dark matter density by the WMAP collaboration. Each of these changes dramatically updates to the 'background' picture of supersymmetry constraints by which a preferred region of that parameter space is recommended. On balance the favored values of $(m_{1/2}, m_0)$ remain relatively small, which bodes well for observation of both proton decay and supersymmetric partners. Finally, we also incorporate a discussion of the alterations which a true string derived context imposes. Indications are that any direct corrections are minimal, yet the identification of a super-unification above the $SU(5) \times U(1)$ scale with the string provides interesting limits on beta function coefficients and super-heavy threshold effects.

2. Heterotic String Model Building

This section describes research in collaboration with Drs. Alon Faraggi, Gerald Cleaver and Dimitri Nanopoulos and Mr. John Perkins, as reported in [15, 16, 17].

In the context of Heterotic string theory there has historically been reasonably good success in the construction of both $MSSM$ and Flipped $SU(5)$ models by the fixing of phases on the fermionic degrees of freedom. In order to produce a low energy effective field theory from a string model, it is necessary to further specify a vacuum state. The Fayet-Illiopoulos anomaly which was originally proposed as a mechanism for electroweak SUSY breaking [3] has been recast here to a dignified position at the string scale. Firstly, it helps greatly to remove the extraneous $U(1)$ factors which tend to appear abundantly in such constructions, and secondly demands the assignment of a non-trivial set of vacuum expectation values (VEVs) which will cancel the offending term such that string-scale SUSY now be preserved!

The door is held open here for a type of phenomenology whereby preservation of supersymmetry is used as the foremost constraint to limit the possible parameter space. In order that the vacuum respect SUSY, all field expectation values must be along so-called 'flat directions', leaving the F- and D-terms of the scalar potential to be zero. It is *from* these required assignments that a beneficial spectral truncation and values for masses and couplings may be realized. We note also in passing that world sheet selection rules form a new criteria for elimination of superpotential terms above simple gauge invariance.

The situation becomes particularly interesting when one attempts to realize flat directions while assigning VEVS to fields transforming under non-Abelian representations of the gauge group. Such a process has been suggested by the insufficiency to date of simpler Abelian-only constructions to generate sufficient mass terms, among other shortcomings. Since the expectation value is now shared among multiple components of a field, satisfaction of flatness becomes an inherently geometrical problem in the group space. Specifically it has been noticed that the D-term $SUSY$ condition appears becomes translated into the imperative that the adjoint space representation of all expectation values form a closed vector sum. Furthermore, the possibility emerges that a single seemingly dangerous F-term might experience a self-cancellation

[3] This approach has generally been supplanted by the writing of 'soft' supersymmetry breaking terms in the Lagrangian which are supposed in turn to descend dynamically from spontaneously broken no-scale supergravity.

among its components. The potential exists that this geometric language can provide an intuitive and immediate recognition of when the D and F conditions are simultaneously compatible, as well as a powerful tool for their comprehensive classification. This is the avenue receiving the greatest attention in the present section, as applied to the cases of $SU(2)$ and $SO(2n)$. These are relevant respectively to previous attempts at reproducing the MSSM and the FNY flipped $SU(5)$ GUT by way of its confining hidden sector element $SU(4) \sim SO(6)$. By necessity, the second case addresses the issues inherent to groups with rank $n > 1$. An additional elimination of some otherwise dangerous terms from the superpotential already been achieved in this formalism. It is hoped that the techniques encountered will be of further benefit in extending the viability of the quasi-realistic phenomenologies already developed.

3. Intersecting D-Brane Model Building

This section describes research in collaboration with Dr. George Kraniotis, Dr. Dimitri Nanopoulos, and fellow students Eric Mayes and Ching-Ming Chen, outlining a complementary path to that demonstrated in [18].

D-branes naturally appear in string theory as static hyper-surfaces on which the ends of open strings terminate when employing the Dirichlet (opposed to Neumann) boundary conditions. With the advent of intersecting D-brane constructions the model-builder has powerful new tools for realizing and interpreting the key ingredients of particle theory. Gauge fields of non-Abelian groups are the massless stringy modes degenerately affixed to 'stacks of these hyper-surfaces. Stacks of N_a such D-branes correspond in a T-dual picture to Chan-Patton degrees of freedom on the worldsheet which are elevated in space-time to a gauged symmetry group $U(N_a)$. Replicated families of bi-fundamentally charged chiral matter arise from the multiple intersections of such stacks. Particle multiplets forming representations under such groups are realized as strings stretched between the two sets of D-branes. At points of intersection, the zero-stretching condition corresponds to (string scale) massless modes as are desired.

The primary goal of this chapter is not to justify or formulate from first principles the properties of intersecting D-branes. Instead we will attempt to summarize in the most systematic and concise manner the content of those rules which emerge from such models, and then formulate procedures for condensing these rules into an efficient extraction of the related gauged symmetry multiplets. We will divide the treatment into two main logical branches. First, a systematic procedural statement of all necessary conditions will be established. The algorithmic appearance of this construction is not accidental. Rather, it represents our initial preferred approach for a numerical treatment of intersecting D-brane model building. At each stage of the discussion, attempts are made to reduce conditions into a format that is more amenable to numerical treatment, and simultaneously easier to digest for the human analyst. Secondly, we will begin to assemble the existing rules into a single unit, investigating what simplifications emerge by taking the union of all conditions rather than applying each in simple disconnected sequence. Our principal reference throughout will be the work of Blumenhagen, Görlich and Ott as recorded in citation [20]. In all situations where specificity is demanded, we will follow this lead in using the $T^6/\mathbb{Z}_4$ orientifold of Type **IIA** string theory.

The simplest space on which to compactify string theory is a torus T^6. However, this flat manifold preserves an excess of supersymmetry, and 'orientifolding' is one possible mechanism to truncate the resulting spectrum to $\mathcal{N} = 1$. The study of $T^6/\mathbb{Z}_4$ has been carefully reviewed, and treated both numerically with proprietary computer programs and analytically in the context of a streamlined and unified statement of the constraints imposed on this model. There are two generic classes of constraint which operate in conjunction to impose strict limits on what gauge multiplets may be realized. Firstly, unbroken supersymmetry is imposed as a condition on the wrapping angles of the D-brane homology cycles. Secondly, there is a Raymond-Raymond charged tadpole term which is necessarily canceled by the *essential* appearance of D-brane stacks wrapping the torus. This is highly analogous to the driving role of FI anomaly in the heterotic string.

So then, the obvious question: What gauge group, and thus what set of stacked D-branes shall be considered? Attempts have been made with reasonable success to construct directly in this way just that most well validated set of fields and interactions, the Standard Model (SM). However, we have argued that precedent and spirit of the SM itself point toward a further unification. However, all argument for a preferred GUT scenario are for naught, if the string will not admit such a gauge group. Available parametric freedom notwithstanding, the string theory is also not infinitely deformable, and it is reasonable to inquire to what classes of models it is most suited.

Flipped $SU(5)$ models have been the principal target in this search. In principle, it could be for example that the flipped variant of $\bar{\mathbf{5}}$ were difficult to produce, or that it never appeared in conjunction with standard Higgs forms. In fact, the truth is quite the opposite. Just as the two $SU(5)$ embeddings each find corollary under $SO(10)$, there is also a perfect parallel to the quantum numbers realized in the orientifolded intersecting D-brane context. Bi-fundamental five-plets are produced at the intersection of a multiplicity-five stack with another single D-brane. These states are accompanied also by an intersection involving either stacks orientifold projection. In the first scenario $U(1)$ charges are summed, and flipped matter results. In the second, the charges are taken in opposition, and the net state carries the markings of standard $SU(5)$. Indeed, it is quite difficult to avoid the appearance of both types of $\bar{\mathbf{5}}$-plets, and we are aware of at least one example in the literature where flipped $SU(5)$ was achieved without intention or realization!

Indeed, a modestly pleasing intermediate result has been achieved and is presented herein. In the **AAA** involution, it appears that Flipped $SU(5)$ does exist with a minimal of extraneous content. However, this is only possible with two rather than 3 generations. Future work is additionally planned along these lines, specifically an improved understanding of the relation between distinct 'complex involutions' is desired. It will also be necessary once the $\mathbb{Z}_4$ scenario is exhausted to expand into other orientifolds. With the presence of a more satisfactory spectrum, it will be possible to begin the hard phenomenology of Yukawa couplings, mass matrices and etc. Finally, it may profitable to return to a numerical treatment, but incorporating the improved and condensed statements of constraint which have been developed.

CHAPTER II

GRAND UNIFICATION AND PROTON DECAY

A. The Call of Unification

1. Supersymmetric Unification

Grand Unified Theories are a logically essential ingredient to the program of reductionism in particle physics. To forward the goal of extracting greater output, both more prolific and more specific, from an ever diminishing set of manually input model parametrization, symmetry considerations are the physicist's greatest tool. Providing both a rigorous notion of charge conservation and a strict representational framework into which all member fields must assemble, such constructions also fully encode the dynamics of interaction when locally covariantly gauged. Having secured the symmetry breaking of $SU(2)_L \times U(1)_Y$ to the $U(1)$ group of electromagnetism within the experimental bag around the 100 GeV scale [1], it is only natural to consider an intermingling of all known interactions. The dynamics of logarithmic renormalization which would govern the energy Q-dependencies of such a suggestion are well defined in terms of the one-loop beta function coefficients b_i that respectively meter the q_i-charged matter content.

$$\frac{1}{\alpha_i(Q)} - \frac{1}{\alpha_{\mathrm{GUT}}} = \frac{b_i}{2\pi} \ln \frac{M_{\mathrm{GUT}}}{Q} \tag{2.1}$$

A survey of this process within the Standard Model (SM) reveals a gentle plodding of the three couplings toward a near crossing at an ultra-high energy of around 10^{15} GeV. However, the Standard Model has its own afflictions to deal with, most notably stabilization of the electroweak scale and an over-abundance of unspecified pa-

[1] Failing only a definitive signal of the Higgs field.

rameters. Supersymmetry (SUSY) proposes to remedy both maladies by the pairwise interrelation of fields transforming in the tensor and spinor Lorentz representations respectively.

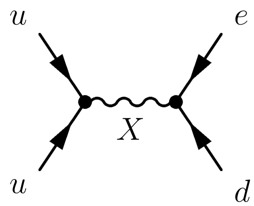
Spontaneously broken symmetries require that only space-time scalars may take nonzero vacuum expectation values if Lorentz invariance is to be preserved. However, scalar fields alone have no protection against bare masses in the action, and are moreover prone to catastrophic quadratic renormalization. Tethering all scalars to a spin-1/2 partner can enforce a transference of beneficial fermionic properties such as a more mild log-type running. Specifically, chiral symmetry may exclude undesirable mass terms, and divergences are greatly softened by finely structured cancellations between partnered Bose diagrams and Fermi loops with their characteristic minus signs. SUSY furthermore offers a guiding principle by which a Lagrangian can be greatly constrained. In this picture, all field masses and Yukawa couplings, as well as the scalar vacuum are fully determined by just a single holomorphic function dubbed the Superpotential.

These attractive features do not come though without a price. First, it is necessary to fully double the known world, as no known 1/2-integer spin-related particles share common quantum numbers suitable for association as supersymmetric partners. The number of Higgs multiplets must also double since up- and down-type masses cannot now arise from a single field once complex conjugation is disallowed in the potential[2]. Furthermore, a mechanism must be posited to shift these unseen super-partners to a suitably invisible mass scale without destroying the initially desired hierarchy stabilization at M_Z. It is also of material interest whether this radical revision of the field content will destroy the picture of GUT convergence. Sugges-

[2]Alternatively considering the spin-1/2 partners, a single field would induce an uncanceled chiral anomaly.

tively, the needed modifications to the b_i instead appear to perfect the unification to within just parts of one thousand. What before gave hope that the idea was at least in the right ballpark is now difficult for even the most cynical to dismiss as pure happenstance. Also interesting is the observation that this more gentle coalescence shifts the GUT scale even further upward by a factor of about ten. This is yet one step closer to the Planck regime, where the gravitational interaction must not be ignored. Indeed though, even this frontier is singularly the province of supersymmetry, which by its unique and consistent intermingling of internal quantum numbers with space-time itself, becomes a theory of general covariance when made local. However, exuberance must be tempered for while yet, as second order effects and the corrections from crossing the light SUSY and heavy GUT mass thresholds threaten to undo our success unless delicately balanced. Additionally, any new predictions generic to GUTs must be evaluated in light of experimental metes and bounds.

Perhaps the most striking maiden feature of Grand Unification is that the notion of strict baryon and lepton number conservation is now abandoned. By placing these formerly distinct entities within common multiplets they are now bound to share interaction vertices. With this development, the stability once granted to the proton as the lightest member of its class is held forfeit. Fortunately, preservation of the universe-as-we-know-it can also be afforded by another means. GUTs are in fact self-protecting in this regard, predicting these exotic interactions be mediated by the broken, i.e. ultra-heavy, gauge Bosons and thus that they are extremely rare. Nevertheless, since the process is forbidden outright within the SM, this offers a particularly prime target for differentiation of theory by experiment. The chief decay mode $p \to e^+\pi^0$ goes like a fourth power of the unified scale. The supersymmetric extension of that characteristic energy translates here to four orders of magnitude in proton lifetime and only by that feint does the predicted rate suffice to evade current detection limits.



$$\Longrightarrow \quad \begin{array}{c} p \to e^+\pi^0,\ \tau_p \propto M_X^4 \\ \mathbf{d\!=\!6} \end{array} \tag{2.2}$$

But what SUSY gives, she can also take away. The pair of electroweak Higgs doublets which that theory requires must also here embed two GUT multiplets, and filling out those representations thus again enlarges the Higgs sector. Mixing is generically allowed between these new fields, or more pertinently their super-partners, which in turn may enable a less noble but more rapid channel of decay. The procedure for elevating these dangerous states to a sufficiently high mass recalls to us the original hierarchy quandary and is not without distinct consequences for low-energy phenomenology even if successful. It is time then to get specific and evaluate all such considerations within the confines of a well-defined model.

2. Selecting a Unified Model

If the notion of a GUT is to be feasible, then one must necessarily inquire as to candidates for the unified group structure and what representational form the known interactions and fields would take within that group. To contain the SM, $SU(3)_C \times SU(2)_L \times U(1)_Y$, which carry respectively two, one and one diagonal generators, a group of minimal rank four is required. The natural starting point is then $SU(5)$, whose lowest order representations are the singlet **1**, the fundamental **5**, the (anti) symmetric tensors **10** and **15**, the adjoint **24** which transforms as the generators, and their related conjugates.

Within each family, all SM states must be assigned a residence which is compatible with their existing quantum numbers. A charge-parity involution is understood where needed such that all grouped fields will carry a consistent handedness. The six quark states of the left-doublet, color-triplet must then fit in at least an anti-symmetric **10**. This leaves four spots open, tidily filled by one right-handed color triplet and a right handed singlet. The remaining color-triplet partners neatly with the electron-neutrino left-doublet as a $\bar{\mathbf{5}}$, canceling the non-Abelian anomaly of the **10**. Not only are the fifteen Standard Model states thus uniquely and compactly represented, but we are gifted additional 'wisdom' in the process. By assignment each of the **10** and $\bar{\mathbf{5}}$ are electrically neutral, thus correlating charge quantization to the number of color degrees of freedom. Furthermore, the apparent masslessness of the neutrinos finds a pleasing justification: there is simply no room remaining in which to house a right-handed component.

$$\begin{array}{c} \begin{pmatrix} u \\ d \end{pmatrix}_L \; ; \; u_L^c \; ; \; d_L^c \\[2em] \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \; ; \; e_L^c \; ; \; \boxed{\nu_L^c} \end{array} \quad \begin{array}{c} \text{JUST} \\ \longrightarrow \\ \text{RIGHT} \end{array} \quad \underset{\bar{\mathbf{5}}}{\begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e \\ \nu_e \end{pmatrix}_L} \; ; \; \underset{\mathbf{10}}{\left(\begin{pmatrix} u \\ d \end{pmatrix}_L \; u_L^c \;\; e_L^c \right)} \; ; \; \underset{\boxed{\mathbf{1}}}{\boxed{\nu_L^c}} \tag{2.3}$$

But experimental evidence cannot any longer allow an agnostic position on neutrino masses. The Super-Kamiokande facility in Japan, which houses $50,000$ metric tonnes of ultra-pure water inside a 40-meter high by 40-meter diameter cylindrical tank faced on all sides by a collection of $13,000$ photomultiplier Čerenkov detectors and shielded beneath $2,700$ meters of earth within the cavity of an old mine, has been diligently studying the problem for many years. By comparing the careful observation of neutrino fluxes with atmospheric origination against the expected detection ratios from known interaction cascades they have borne convincing witness to oscillation between the ν_μ and ν_τ sectors [3]. This phenomenon may only occur when the related states carry non-equivalent masses, of which at least one must then be non-zero. However, chirality can only be an invariant quantum number for massless states, and we are thus compelled to introduce a sixteenth element for accommodation of the right-handed neutrino within any considered GUT. It is certainly possible to imagine the new state as a simple singlet outside the main representations already laid down. Surely though it is presumptive to suppose that this right-handed neutrino, while arriving 'last' must also be seated as the 'least'. Every existing position must instead be subject to reassignment. There is indeed then another way, if one is willing to sacrifice charge quantization. We can choose to 'flip' the right-handed quarks placed inside the $\bar{\mathbf{5}}$ and $\mathbf{10}$ while also swapping e^c_L for ν^c_L footnoteFor a comprehensive recent review of flipped $SU(5)$, please consult [21].

$$f_{\bar{\mathbf{5}}} = \begin{pmatrix} u^c_1 \\ u^c_2 \\ u^c_3 \\ e \\ \nu_e \end{pmatrix}_L \quad ; \quad F_{\mathbf{10}} = \left(\begin{pmatrix} u \\ d \end{pmatrix}_L \quad d^c_L \quad \nu^c_L \right) \quad ; \quad l_{\mathbf{1}} = e^c_L \tag{2.4}$$

The cost of this flipping is nothing less than the loss of grand unification, as the resulting symmetry group is enlarged to the non-simple variant $SU(5) \times U(1)_X$. As demonstrated in Fig. 1, the hypercharge does not descend out of $SU(5)$ together

[3]Likewise, $\nu_e \leftrightarrow \nu_\mu$ oscillation has been induced from the observation of solar neutrinos by the SNO collaboration.

with the nuclear forces, but is instead of an admixture of $U(1)_X$ together with the additional U(1) factor which is emergent from that breaking. If this tune sounds familiar though, it is simply a reprise of the theme played out some fourteen orders of magnitude below in the Weinberg-Salam $SU(2)_L \times U(1)_Y$ electroweak 'unification'. Just as it was there no loss to save a true convergence for the future inclusion of color perhaps it is here folly to imagine a full unification which occurs on the border of the Planck mass without waiting on gravity. Just as only experiment could there reject the truly unified but dysfunctional $SU(2)$ model of Georgi and Glashow, between (or against!) these contenders we can again only let phenomenology decide. And here there is no finer judge than the consideration of proton decay.

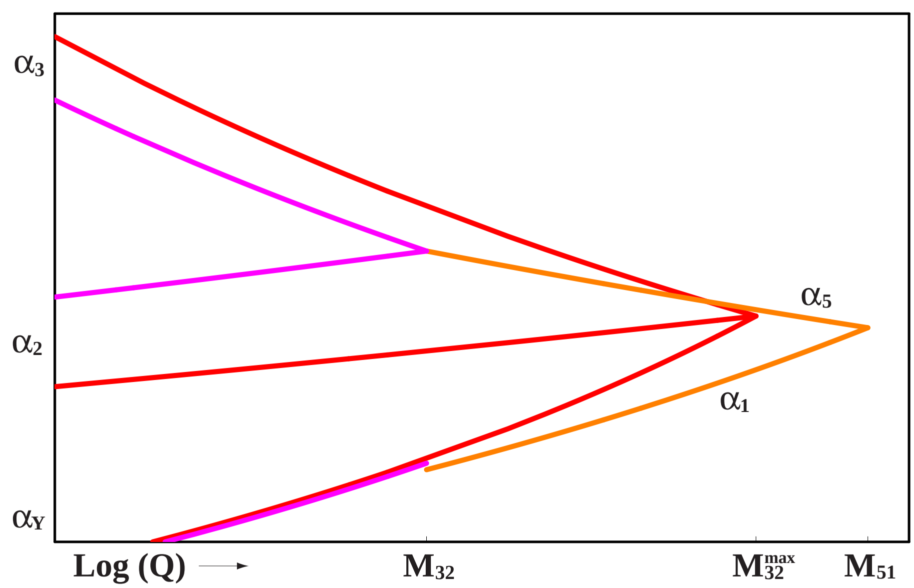
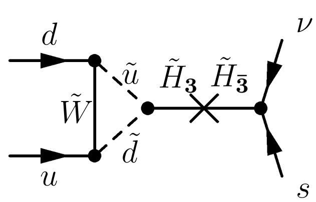


Fig. 1. *A Heuristic Demonstration of Flipped Gauge Coupling Unification. Notice the discontinuity in the (purple) line of $U(1)_Y$, as it remixes between the "grand unified" $U(1)$, and that which emerges out of broken $SU(5)$ at the scale M_{32}. Proceeding upward from this interim stage in orange, $SU(5) \times U(1)$ is itself unified at some higher scale M_{51}. For comparison, the standard $SU(5)$ scenario is shown in red with a single unification at $M_{32}^{\rm max} \geq M_{32}$, and predicting a larger value for $\alpha_s(M_Z)$.*

B. The Consequence of Unification

1. Instability of the Proton

Despite the incredible contribution of knowledge made by Super-Kamiokande to the physics of neutrino oscillations, this is in fact not even the principal task of that experiment. Faced with the incomprehensibly long time scales on which proton decay is expected to be manifest, some $19-26$ orders of magnitude older than the universe itself, we can only hope to observe this process within some reasonably finite interval by leveraging Avogadro's number to our benefit and watching some very large number of nuclei simultaneously. Indeed, that is precisely the role of the extravagant size allowed to this detector. Turning first to the $p \to K^+\bar{\nu}$ partial lifetime as mediated by the triplet higgsinos of $SU(5)$, a lower bound of 6.7×10^{32} years has been established at the 90% confidence level.



$$\Longrightarrow \qquad p \to K^+\bar{\nu},\ \tau_p \propto M^2_{\tilde{H}_{\mathbf{3}}} \qquad \mathbf{d\!=\!5} \tag{2.5}$$

This is a so-called dimension-five decay, summing the mass level of the two-boson, two-fermion effective vertex *à la* Fermi. The upper bound on its rate translates directly to a minimal mass for the color-triplet Higgs of around 10^{17}GeV [22]. Conversely though, compatibility of a strict unification with the precision LEP measurements of SM parameters at M_Z places a lower limit on this mass of order 10^{15}GeV.

Flipped $SU(5)$ evades this incongruity by means of the 'missing-partner mechanism' [23, 24, 25] which naturally splits the heavy triplets $H_{\mathbf{3}}$ within the five of Higgs (h) away from the light electroweak components $H_{\mathbf{2}}$. Specifically, since the flipped $\mathbf{10}$ now contains a neutral element it is possible to allow vacuum expectation values for the breaking of $SU(5)$ to arise within a Higgs decaplet H from this representation. The GUT superpotential elements HHh and $\bar{H}\bar{H}\bar{h}$ then provide for the mass terms $\langle\nu^c_H\rangle d^c_H H_{\mathbf{3}}$ (and conjugate), while $H_{\mathbf{2}}$ is left light, having no partner in H with which to make a neutral pairing. So then is adroitly bypassed all insinuation of a hand-built term $Mh_{\mathbf{5}}h_{\bar{\mathbf{5}}}$ to finely tune against the putative adjoint GUT Higgs $h_{\mathbf{5}}h_{\bar{\mathbf{5}}}\Sigma_{\mathbf{24}}$ for fulfillment of this same goal. And with that term goes also the undesirable triplet mixing, the dangerously fast proton decay channel and the fatal limits on the mass of $H_{\mathbf{3}}$ [4]. As for standard $SU(5)$ however, this is just another splash of cold water from our friends at Super-K. And as we have mentioned, they have no shortage of cold water.

This diversion put aside, the dimension six decay $p \to e^+\pi^0$ may now regain our attention. With aid of the SUSY extension, neither theory is under any fear from the current lower bound of 1.6×10^{33} years for this mode. That is not to say though that interesting differences do not exist between the pictures. In standard $SU(5)$ there are two effective operators which contribute in sum to this rate. The first vertex arises from the term $\mathbf{10}\,\bar{\mathbf{5}}\,\mathbf{10}^*\,\bar{\mathbf{5}}^*$ and the second from $\mathbf{10}\,\mathbf{10}\,\mathbf{10}^*\,\mathbf{10}^*$ with a relative strength of $(1 + |V_{ud}|^2)^2$. However, in flipped $SU(5)$, e^c_L no longer resides within the $\mathbf{10}$, so the positronic channel makes use of only e^c_R decays utilizing the operator which contains the representation $\bar{\mathbf{5}}$. Taking the central value of .9738(5) for the Cabibbo quark-mixing phase V_{ud} leads to a suppression of the total rate by a factor of about five after dividing out the correction $(1+(1+|V_{ud}|^2)^2)$. In opposition to this effect is a tendency toward more rapid decay due to dependence on the intermediate partial unification scale M_{32} rather than the traditional GUT value. In fact, we will see

[4]The $d = 5$ mode is not *entirely* abandoned, as there does remain the supersymmetric term $\mu h\bar{h}$ for suppression of electroweak axions. It is slower though by a ratio $(\frac{\mu}{M_{H_{\mathbf{3}}}})^2$, where $\mu \sim$ TeV.

that this second distinction generally overwhelms the first, leading on balance to a net shorter prediction of the proton lifetime in flipped $SU(5)$. This is a beneficial result in light of the next generation proposals [26] for super-massive water-Čerenkov detectors weighing up to 10^6 tonnes. Sampling a number of nuclei greater by some factor of twenty than Super-K, such an experiment could be sensitive to $\tau(p \to e^+\pi^0)$ at a level of around 10^{35} y.

Lower bounds can only ever exclude a model, never truly supporting any one competing suggestion. The real goal of course is to constrain this number from both directions, and assisted by the shorter net flipped $SU(5)$ lifetime, the reach of next-generation experiments could come tanalizingly close to probing the most relevant portions of parameter space. However, as we will see, even this great step may unfortunately be insufficient. Following the results of [27, 22] and references therein, we present a numerically parametrized expression for the desired lifetime, with coefficients appropriate to the flipped specialization already absorbed.

$$\tau(p \to e^+\pi^0) \;=\; 3.8 \times 10^{35} \left(\frac{M_{32}}{10^{16}\mathrm{GeV}}\right)^4 \left(\frac{\alpha_5(M_{32}^{\max})}{\alpha_5(M_{32})}\right)^2 \left(\frac{0.015\mathrm{GeV}^3}{\alpha}\right)^2 \;\mathrm{y} \qquad (2.6)$$

The amplitude for this process is proportional to the overall proton wave function normalization at the origin. The reduced hadronic matrix elements

$$\langle 0|\epsilon_{ijk}(u^i d^j)_R u_L^k|p(\mathbf{k})\rangle \;\equiv\; \alpha\, \mathrm{u}_L(\mathbf{k}), \qquad (2.7)$$

$$\langle 0|\epsilon_{ijk}(u^i d^j)_L u_L^k|p(\mathbf{k})\rangle \;\equiv\; \beta\, \mathrm{u}_L(\mathbf{k})\,, \qquad (2.8)$$

have recently been updated to the central reference values $\alpha = \beta = 0.015\mathrm{GeV}^3$ by the JLQCD collaboration using lattice methods. Somewhat higher than previous estimates, their calculation has also greatly bolstered the prospects for observability in this mode. The uncertainty which accompanies these factors into (2.8) stands at about 20%.

We close this section with a survey of some characteristic predictions from flipped $SU(5)$ proton decay based on the baryon-number violating effective potential [28, 29]:

$$\begin{aligned}\bar{\mathcal{L}}_{\Delta B\neq 0} &= \frac{g_5^2}{2M_{32}^2}\big[(\epsilon^{ijk}\bar{d}_k^c e^{2i\eta_{11}}\gamma^\mu P_L d_j)(u_i\gamma_\mu P_L\nu_L) + h.c. \\ &+ (\epsilon^{ijk}(\bar{d}_k^c e^{2i\eta_{11}}\cos\theta_c + \bar{s}_k^c e^{2i\eta_{21}}\sin\theta_c)\gamma^\mu P_L u_j)(u_i\gamma_\mu P_L\ell_L) + h.c.\big] \quad (2.9)\end{aligned}$$

where θ_c is the Cabibbo angle. Unknown parameters in (2.9) are the the CP-violating phases $\eta_{11,21}$ and lepton flavor eigenstates ν_L and ℓ_L related to the mass diagonal mixtures as:

$$\nu_L = \nu_F U_\nu \quad , \quad \ell_L = \ell_F U_\ell \, . \qquad (2.10)$$

These mixing matrices $U_(\nu,\ell)$ take on added currency in the age of neutrino oscillations. Having seen there evidence for near-maximal mixing, it seems reasonable to suspect that at least some e/μ entries are also $\mathcal{O}(1)$ in U_ℓ. From this point it will indeed be assumed that $|U_{\ell_{11,12}}|^2$ are $\mathcal{O}(1)$, thus avoiding further large numerical suppressions of both the $p \to (e/\mu^+)\pi^0$ rates [5]. No more can be said though regarding the ratio of $p \to e^+X$ and $p \to \mu^+X$ decays, and as such it would be good that any next-generation detector be equally adept at the exposure of either mode [6]. Despite these ignorances, it can be robustly stated that [30]:

$$\begin{aligned}\Gamma(p\to e^+\pi^o) &= \frac{\cos^2\theta_c}{2}|U_{\ell_{11}}|^2\Gamma(p\to\bar{\nu}\pi^+) = \cos^2\theta_c|U_{\ell_{11}}|^2\Gamma(n\to\bar{\nu}\pi^o) \\ \Gamma(n\to e^+\pi^-) &= 2\Gamma(p\to e^+\pi^o) \quad , \quad \Gamma(n\to\mu^+\pi^-) = 2\Gamma(p\to\mu^+\pi^o) \\ \Gamma(p\to\mu^+\pi^o) &= \frac{\cos^2\theta_c}{2}|U_{\ell_{12}}|^2\Gamma(p\to\bar{\nu}\pi^+) = \cos^2\theta_c|U_{\ell_{12}}|^2\Gamma(n\to\bar{\nu}\pi^o) \qquad (2.11)\end{aligned}$$

[5] Note that there is no corresponding suppression of the $p \to \bar{\nu}\pi^+$ and $n \to \bar{\nu}\pi^0$ modes, since all neutrino flavors are summed over.

[6] Hereafter, whenever is mentioned the decay process $p \to e^+\pi^0$, it is also taken to include the muonic product ambiguity.

We note [30, 27] that the flipped $SU(5)$ predictions for decay ratios involving strange particles, neutrinos and charged leptons differ substantially from those of conventional $SU(5)$. Comparison of such characteristic signals then constitutes a potentially powerful tool for establishing mixing patterns and differentiating between GUT proposals.

2. The Renormalization Group

So then, we have reached something of an impasse in terms of garnering direct information from existing proton decay limits. However, a quick look at (2.6) reveals a dependence on just a single parameter germane to flipped $SU(5)$: the intermediate mass scale M_{32}. This scale is the most essential defining characteristic of the deviation from a strict unification which is expected once the electric charge is subdivided between multiple groups. It is also phenomenologically quite rich, both in terms of predictivity and predictability. The agenda thus now becomes a description of what factors influence and constrain the mass M_{32}, sometimes then turning the analysis around as a prediction for the proton lifetime.

Specializing (2.1) to the case at hand, we have [27]:

$$\frac{1}{\alpha_Y} - \frac{1}{\alpha'_1} = \frac{b_Y}{2\pi} \ln \frac{M_{32}}{M_Z} \tag{2.12}$$

$$\frac{1}{\alpha_2} - \frac{1}{\alpha_5} = \frac{b_2}{2\pi} \ln \frac{M_{32}}{M_Z} \tag{2.13}$$

$$\frac{1}{\alpha_3} - \frac{1}{\alpha_5} = \frac{b_3}{2\pi} \ln \frac{M_{32}}{M_Z} \tag{2.14}$$

As usual, $\alpha_Y = \frac{5}{3}(\alpha_{em}(M_Z)/\cos^2\theta_W)$, $\alpha_2 = \alpha_{em}(M_Z)/\sin^2\theta_W$, $\alpha_3 \equiv \alpha_s(M_Z)$, and the SUSY one-loop beta function coefficients are $b_Y = 33/5$, $b_2 = +1$ and $b_3 = -3$. The limit of a strict triple-unification occurs whenever α'_1 (the hypercharge evaluated at M_{32}) exactly matches α_5, toward which the nuclear couplings are running. Enforcing this condition *à la* conventional $SU(5)$, we take the linear combination of (2.12, 2.13, 2.14) which will eliminate both of the scale dependent terms α_5 and $\sin^2\theta_W$ [7], solving specifically for the 'maximal' allowed value of M_{32}.

$$M_{32}^{\max} = M_Z \times \exp\left\{\frac{\pi(3\alpha_s - 8\alpha_{em})}{30\alpha_{em}\alpha_s}\right\} \tag{2.15}$$

Similarly excluding α_5 now between just (2.13,2.14), and using (2.15) to replace M_Z via the expansion

$$\ln\frac{M_{32}}{M_Z} = \ln\frac{M_{32}}{M_{32}^{\max}} + \ln\frac{M_{32}^{\max}}{M_Z}\,, \tag{2.16}$$

one arrives at the following most useful expression [8]:

$$\alpha_s(M_Z) = \frac{\frac{7}{3}\,\alpha_{em}}{5\sin^2\theta_W - 1 + \frac{10}{\pi}\,\alpha_{em}\ln\left(M_{32}^{\max}/M_{32}\right)} \tag{2.17}$$

We can then isolate α_5 from (2.14), using again (2.15,2.16).

$$\alpha_5 = \left[\frac{3}{5\alpha_s} + \frac{3}{20\alpha_{em}} - \frac{3}{2\pi}\ln\left(\frac{M_{32}^{\max}}{M_{32}}\right)\right]^{-1} \tag{2.18}$$

Evaluating M_{32} at its extremal value, this reduces to:

$$\alpha_5^{\max} \equiv \alpha_5(M_{32}^{\max}) = \frac{20\,\alpha_{em}\alpha_s}{3(\alpha_s + 4\alpha_{em})} \tag{2.19}$$

We may gauge the consistency of this picture by comparing (2.17) against recent precision electroweak data as compiled by the Particle Data Group [31].

$$\begin{array}{rlcl} \alpha_{em}(M_Z) = \frac{1}{127.918\pm.018} & , & \alpha_s(M_Z) = .1187 \pm .0020 \\ \sin^2\theta_W^{\overline{MS}}(M_Z) = .23120 \pm .00015 & , & M_Z = 91.1876 \pm .0021\text{GeV} \end{array} \tag{2.20}$$

As advertised, the strict unification limit $M_{32} \to M_{32}^{\max}$ is validated to a surprising accuracy.

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[7] The sense in which $\sin^2\theta_W$ is in this context a function of the generic scale M_{32} rather than an experimentally input number will become clear shortly.

[8] The coefficient $\frac{10}{\pi}$ shown here corrects an erroneous value appearing in some earlier publications.

However, things are not quite so simple as has been suggested. More specifically, there are corrections at the next order due to two-loop effects and the detailed crossing of mass thresholds at both the light supersymmetric and heavy grand-unified scales. The sum of these contributions may be effectively accommodated via the substitution

$$\sin^2\theta_W \rightarrow \sin^2\theta_W - \delta_{\rm 2loop} - \delta_{\rm light} - \delta_{\rm heavy} \tag{2.21}$$

in (2.17). It is worth emphasizing that although each of these corrections is most properly associated with shifts in the beta coefficients, it has been instead convenient to parametrize the bulk of our ignorance by just (2.21), locking each of the b_i at the prior quotes. So then, if $\alpha_s(\sin^2\theta_W, M_{32})$ from (2.17) is forced to the central measured value, we may best instead interpret that equation to express the necessary trade offs between the unknown dependent thresholds (as contained inside $\sin^2\theta_W$) and the ratio $M_{32}/M_{32}^{\rm max}$ [32]. This stipulation now explains the great care taken to ensure that $\sin^2\theta_w(M_{32})$, and similarly $\alpha_5(M_{32})$, are never equated across expressions evaluated at different specializations of this mass. Of course, when it comes time to plot renormalization curves, we are bound to use the experimentally established $\sin^2\theta_W$, as the weak coupling constant $\alpha_2(M_Z)$ is well known. To do so consistently, we must transfer the two-loop and threshold corrections back where they really belong, i.e. into numerical shifts of the b_i.

The two-loop shift, $\delta_{\rm 2loop} \approx 0.0030$, is calculable, although $\delta_{\rm light}$ and $\delta_{\rm heavy}$ are undetermined at this stage and can in principle carry either sign. Neglecting these thresholds for the moment, $\delta_{\rm 2loop}$ alone lifts the ‘prediction’ for α_s in standard $SU(5)$ to around 0.130 [33, 34, 35, 36, 37, 38]. Restoring a value within one standard deviation of the present central value requires the non-trivial assistance of $\delta_{\rm light}$ and/or $\delta_{\rm heavy}$ such that the net correction of (2.21) is suppressed. However, $\delta_{\rm light}$ is too small or even of the wrong sign in large portions of parameter space. Moreover, as pointed out in [39, 40], appealing here to the aid of δ_{heavy} may only compound the conflicts of $SU(5)$ with proton decay limits [41, 42] for the $\tau(p \to \bar{\nu} K^+)$ mode.

In flipped $SU(5)$ though, there is an alternate way to restore $\alpha_s(M_Z)$ to a satisfactory range. Indeed, the full offset of the two-loop correction can be countered by simply setting $M_{32} \approx \frac{1}{2} M_{32}^{\text{max}}$ in (2.17). This correlation between M_{32} and $\alpha_s(M_Z)$ is explored graphically in Fig. 2. The curve demonstrates a strong preference for the reduced scale partial unification of flipped $SU(5)$ relative to the bounding value of $M_{32}^{\text{max}} = 2.23 \times 10^{16}$ GeV. Also depicted are a number of the proposed benchmark CMSSM scenarios [43] (now updated to the post-WMAP era as denoted with primed characters [44]) which attempt to survey the phenomenologically preferred SUSY parameter space [9]. They are restricted to $A_0 = 0$, but otherwise span the viable ranges of $m_{1/2}, m_0$ and $\tan\beta$, and represent both signs for μ. The shift δ_{light} is also included for these alphabetically labeled points, plotted at the central value of α_s [10]. Although the generic tendency of that effect here is to soften the scale reduction, there exists no benchmark model for which a conventional unification is consistent with the measured values of $\alpha_s(M_Z)$ and $\sin^2\theta_W$ unless one also invokes action of the GUT thresholds. We note that the quoted error in $\alpha_s(M_Z)$ is about 0.002, which translates to an uncertainty in M_{32} of order 20%, and in turn an induced uncertainty in the proton lifetime of a factor around two. The error associated with the uncertainty in $\sin^2\theta_W$ is somewhat smaller.

[9] The benchmark point F' from the focus point region at large m_0 cannot provide for electroweak symmetry breaking, taking the recently enlarged top quark mass into our analysis. It is thus dropped here from further consideration.

[10] The prediction for M_{32} at each benchmark for variations in α_s follows a curve like that shown, but passing through the point in question.

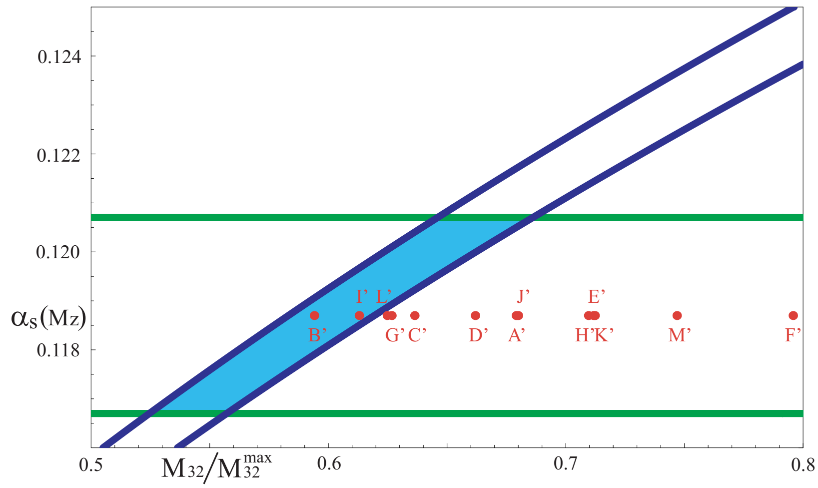

Fig. 2. $\alpha_s(M_Z)$ *vs.* $M_{32}/M_{32}^{\max}$ *with Benchmarks. The solid blue curved lines indicate the range allowed to* M_{32} *for a given value of* $\alpha_s(M_Z)$*, within one standard deviation of* $\sin^2\theta_W$*. The two-loop correction is implemented to the exclusion of either threshold. The green horizontal bars depict one standard deviation in* α_s*. The labeled points supplement these considerations with the value of* δ_{light} *appropriate to each CMSSM benchmark.*

C. The Completion of Unification

1. The Supersymmetric Threshold

We turn now to a detailed discussion of the light threshold effects. Although the theoretical motivations and phenomenological necessities of low-energy supersymmetry are myriad, all experiments have yet to directly observe the signal of any superpartner field. We are not disheartened though, as even this fact is only yet another clue into the nature of how SUSY must be broken. And moreover, the extreme interdependencies imposed by this symmetry allow concrete predictions to be made for the unseen sector. It is essential that any proposal for splitting masses at the infrared limit do so without disrupting the cancellation of ultraviolet divergences or unmaking the 100 GeV electroweak hierarchy. The most general approach to parameterizing the needed SUSY deviation thus involves an insertion into the fundamental action of non-symmetric terms with mass-dimension less than or equal to three in the fields. These 'soft-breaking' expressions come in four basic varieties, proportional to existing SUSY-valid terms but bearing non-canonical coefficients. Namely, they are the spin-$\frac{1}{2}$ gaugino masses $m^i_{1/2}$, the scalar mass-square matrix $(m_0^2)^{ij}$, and the bi- and tri-linear holomorphic superpotential rescaling B_0^{ij} and A_0^{ijk} [11].

The suggestion of the CMSSM is that each class of correction be assigned a universal value at the GUT scale. As a practical matter, it is enough that without such a simplification we would be unable to sensibly proceed. We may take comfort though that this democratic scenario can naturally emerge in the effective potential of a spontaneously broken no-scale supergravity. Failing exclusory evidence, no 'more sophisticated' choice has at any rate a superior motivation. From the enlarged SUSY Higgs sector there is also the (supersymmetry conserving) mixing parameter μ as needed to prevent a massless state H_i, and the ratio $\tan\beta$ of expectation values $\langle H_2\rangle/\langle H_1\rangle$. Although the sum-squared of these VEVs is predicted just as in the SM, their ratio is undetermined and can be used to partially alleviate from the Yukawa couplings the onus of providing an up/down mass disparity. Enforcing a radiative trigger for the electroweak symmetry breaking [12] is sufficient to further eliminate the B_0-coefficient and the absolute value of μ from consideration. This leaves then just the short list $m_0, m_{1/2}, A_0, \tan\beta$ and the sign of μ, from which the entire Higgs [13] and

[11]The parameters B_0^{ij} also influences predicted masses, while A_0^{ijk} are related to the Yukawa couplings.

[12]We recall that the large valued Yukawa necessary to drive the mass-squared of the up-type Higgs negative around M_Z successfully predicted a top quark mass above 100 GeV in an era when it was generally supposed to lie in the mid-teens.

[13]The MSSM requires 8 Higgs degrees of freedom arranged as two pair of complex scalar doublets. Three Goldstone components are absorbed by the newly massive

s-particle spectrum can be calculated[14]. Insomuch as these masses are known, the light threshold correction is also known, taken approximately as [39, 40, 32]

$$\begin{aligned}\delta_{\text{light}} &= \frac{\alpha}{20\pi}\Big[-3L(m_t) + \frac{28}{3}L(m_{\tilde{g}}) - \frac{32}{3}L(m_{\tilde{w}}) - L(m_h) - 4L(m_H) \\ &\quad + \frac{5}{2}L(m_{\tilde{q}}) - 3L(m_{\tilde{\ell}_L}) + 2L(m_{\tilde{\ell}_R}) - \frac{35}{36}L(m_{\tilde{t}_2}) - \frac{19}{36}L(m_{\tilde{t}_1})\Big] \;, \qquad (2.22)\end{aligned}$$

where $L(x) \equiv \ln(x/M_Z)$. Ignoring δ_{heavy} for the time being, this translates directly into the required compensatory value of M_{32} from (2.17), and thus also to a prediction for the flipped $SU(5)$ proton lifetime, (2.6).

The strongest dependencies lie with the universal gaugino and scalar masses $(m_{1/2}, m_0)$, so these are generally plotted in opposition as in Figs. 3 [45], for fixed values of the other three parameters. As is the case here, the tri-linear coupling A_0 is most often set to zero. Within the expanse of these planes is expressed the bounty of a great catalog of experiments. Interestingly, one of the most confining of these observations comes from cosmology, principally via the WMAP satellite collaboration [46, 47], which has measured with great precision the neutral non-baryonic cold dark matter contribution to the energy density of the universe.

$$0.094 < \Omega_{CDM} h^2 < 0.129 \qquad (2.23)$$

SUSY offers the neutralino χ, a mixture of the uncharged Higgs and gauge super-partners, as a natural candidate for the source of this remnant matter. Predicted in large portions of parameter space to comprise the lightest supersymmetric particle (LSP), the neutralino is thus afforded stability under conservation of R parity.

$(W^\pm, Z^0)$, while five components $(h^0, A^0, H^\pm, H^0)$ remain observable.

[14]More specifically, the baseline from which the paired super-partners split is established. By definition, the observed particles are the lighter, and from their measured masses the heavier are then deduced.

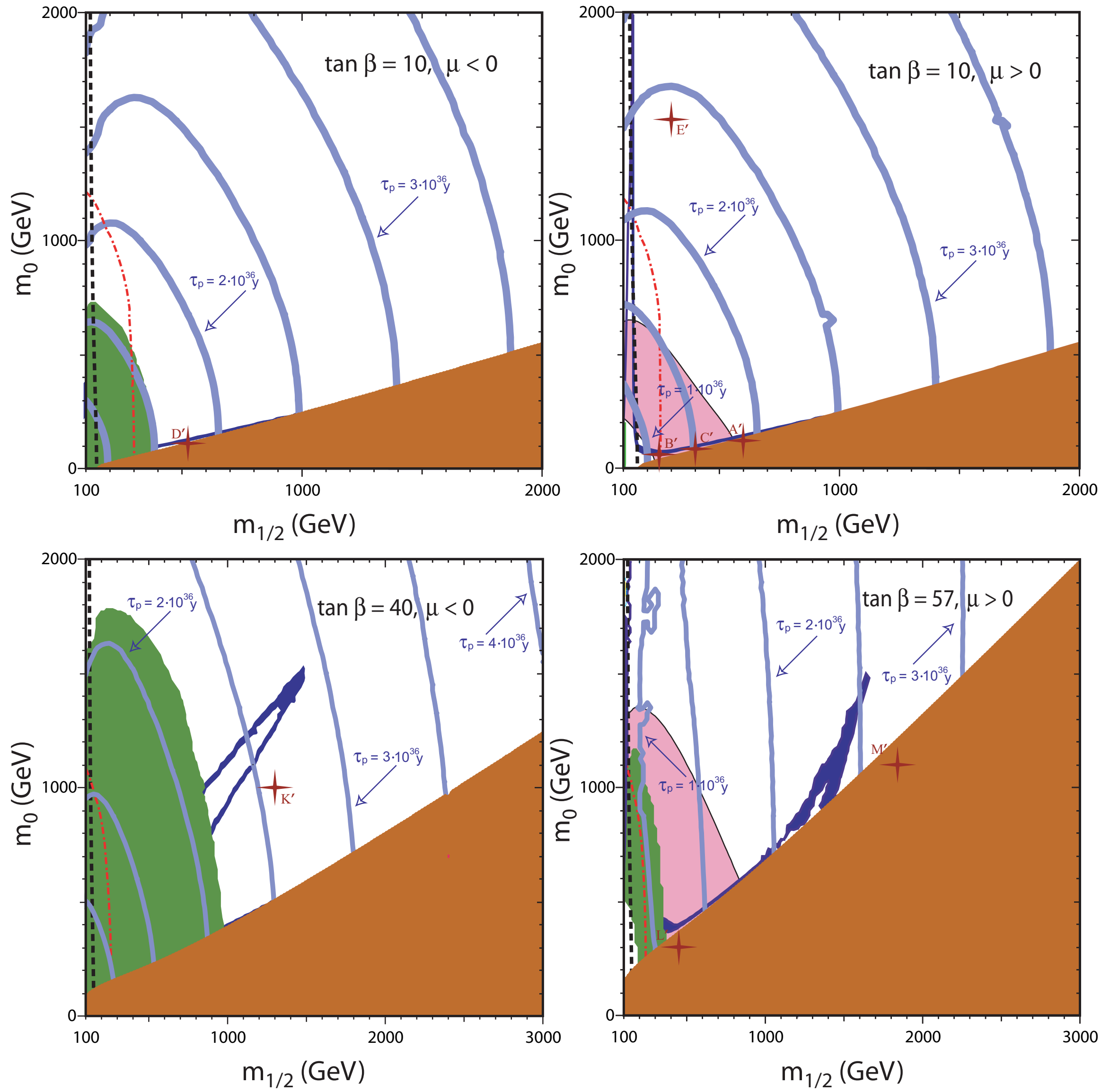

Fig. 3. *Proton Lifetimes within the CMSSM* $(m_{1/2}, m_0)$ *Plane. The solid (light blue) lines are contours of* $\tau(p \to e/\mu^+\pi^0)$ *as dependent on the locally determined light threshold corrections. Benchmark points with the corresponding value of* $\tan\beta$ *and sign of* μ *[32] are marked with stars where applicable. This data is overlaid on the charts of [45], which map the suitability of the contending parameter space according to satisfactory amounts of cold dark matter (deep blue), presence of a charged LSP (orange), overlarge* $b \to s\gamma$ *corrections (green), consistency with* $g_\mu - 2$ *(pink), and LEP lower limits on the lightest Higgs (red dash-dot) and* $\chi^\pm$ *(thickly dashed black) masses.*

There are however modes for bulk pairwise $\chi - \chi$ annihilations in the comparatively lower $(m_{1/2}, m_0)$ regions, and also the more exotic variants of $\chi - \tilde{\ell}$ co-annihilation in the large $m_{1/2}$ 'tail', resonant 'funnel' direct-channel annihilations via the heavier A, H Higgs at large $(m_{1/2}, m_0)$, and the 'focus-point' [15] modes at large m_0. For each process only a very slender margin of the $(m_{1/2}, m_0)$ plane (marked in deep blue) can retain a satisfactory neutralino density to constitute the CDM without over-closing the universe. Prominent at lower values of m_0, and sloping upwardly along the co-annihilation tail, the solidly shaded (deep orange) patch is excluded by prediction of an electrically charged LSP [16].

Another most important marker is the predicted mass of the (lightest) Higgs h. It should not go unnoticed that while the SM allows the Higgs a great variation, SUSY is quite specific and generically states $m_h < M_Z$ so long as the symmetry remains intact. The heaviness of the top quark suggests however that radiative corrections may be quite large, and broken SUSY can permit values less than about 135 GeV. Observation within this range would be a great boon for the theory. Indeed, by taking the recent upward shift of the m_t world average to 177.9 ± 4.4 GeV as courtesy of DØ, the central fit for m_h narrowly out-paces the current experimental lower limit of 114.4 GeV at 95% C.L., landing right on top of the still disputed $115^{+1.3}_{-0.9}$ GeV signal from LEP's final year (2000) of operation. Lower accelerator bounds on m_h and ($m_{\chi^\pm} \geq 103.5$ GeV) are depicted in turn by the (red) dot-dashed and heavy vertical (black) dashed lines at small $m_{1/2}$ [17].

[15] This already less pleasing scenario is cast into a worse light by a recent upsurge of the central top quark mass.

[16] As partner to the heaviest lepton, the stau $\tilde{\tau}$ is the lightest slepton.

[17] These curves are placed as calculated using the `FeynHiggs` code [48, 49], and some uncertainty exists particularly for the m_h contour. For discussion of the implementations of these constraints with and without `ISASUGRA`, see [44, 50].

The curving (pink) strips shown for $\mu > 0$ are inclusive limits from the also controversial BNL E821 measurement of the 'anomalous-anomalous' magnetic moment of the muon, i.e. surplus contributions to $(g_\mu - 2)$ as induced by supersymmetric graphs. Drawn at the $2-\sigma$ level, they impose a principally upper-bound on the SUSY spectrum such that it is sufficiently light to supplement the Standard Model e^+e^- annihilation contribution (neglecting data from τ decays). The irregular (green) shaded swathes, which take prominence for $\mu < 0$ at small $m_{1/2}$, are excluded by $b \rightarrow s\gamma$. By contrast, they essentially serve as lower limits on SUSY, such that it not disrupt the already successful SM decay predictions. Benchmark [44] locations are marked by stars where appropriate. Finally, we mention as a general warning that regions too far toward the North-Westerly corner of such planes can further suffer difficulty with electroweak symmetry breaking, while the extreme South-West is susceptible to tachyonic spectra.

Over this rich background we have laid topographical contours (light blue) for the predicted variation of the $\tau(p \rightarrow e^+\pi^0)$ proton lifetime within the SUSY parameter space. The program `ISASUGRA` [51], version `7.71`, was used to produce the sparticle spectra on a densely packed [18] sample grid for each plot, assuming $m_t = 177.9$ GeV [19]. To evaluate (2.22), $m_{\tilde{w}}$ (m_H) $(m_{\tilde{q}})$ $(m_{\tilde{\ell}})$ were interpreted respectively as the geometric mean of the chargino and neutralino $(H, A, H^\pm)$ $(\tilde{u}, \tilde{d}, \tilde{s}, \tilde{c})$ $(\tilde{e}, \tilde{\mu})$ masses. The generally lighter mixings of $\tilde{\tau}, \tilde{b}$ and $\tilde{t}$ were each considered separately. We note from Figs. 3 that the 'bulk' regions of the astrophysically preferred parameter space at relatively small values of $(m_{1/2}, m_0)$ generally correspond to $\tau(p \rightarrow e^+\pi^0) \sim 1 \times 10^{36}$ y. However, this territory is often disfavored by either the bounds on m_h and/or $b \rightarrow s\gamma$ decay.

[18] The horizontal spacing between points sampled was comparable to the thickness of these lines.

[19] Heavy singlet neutrinos were not used in the renormalization-group equations.

There are techniques though by which we can systematically rank the goodness of each proposed position, taking a weighted simultaneous consideration of all relevant constraints.

We follow here the efforts of [52], wherein the dimensionality of the the $(m_{1/2}, m_0)$ plane is effectively reduced by considering only points along the preferred WMAP 'strips'. Each sampled CMSSM location is then evaluated under a χ^2 fit

$$\chi^2 \equiv \sum_{n=1}^{N} \left(\frac{R_n^{exp} - R_n^{theo}}{\sigma_n} \right)^2 , \tag{2.24}$$

where σ_n represents the combined error from experiment, parametric uncertainties, and higher order corrections for the n^{th} observable. Here $N = 4$, taking into account the constraints from $M_W, \sin^2\theta_{eff}, (g-2)_\mu$ and $BR(b \to s\gamma)$, and also rejecting all points which violate Higgs or chargino mass limits. This analysis also extends the previous considerations by allowing non-zero values of the tri-linear coupling A_0. We have translated their results into the language of corrections at the light threshold and thus predictions for the proton lifetime as displayed in Figs. 4, following the same procedures discussed previously.

Of the available scenarios studied, we note that the best general fit is obtained in the neighborhood of

$$\begin{aligned} A_0 = -m_{1/2} \quad &, \quad \tan\beta = 10 \\ m_{1/2} \sim 290 \quad &, \quad m_0 \sim 60 \end{aligned} \tag{2.25}$$

with $\mu > 0$, yielding a pleasing net χ^2 deviation of around .292 . The resulting predictions for this case, neglecting δ_{heavy}, are:

$$\delta_{light} \sim -.000170 \quad , \quad \tau_p \sim 1.38 \times 10^{36} y \tag{2.26}$$

This corresponds most closely to the benchmark scenarios (B', C'). With δ_{light} com-

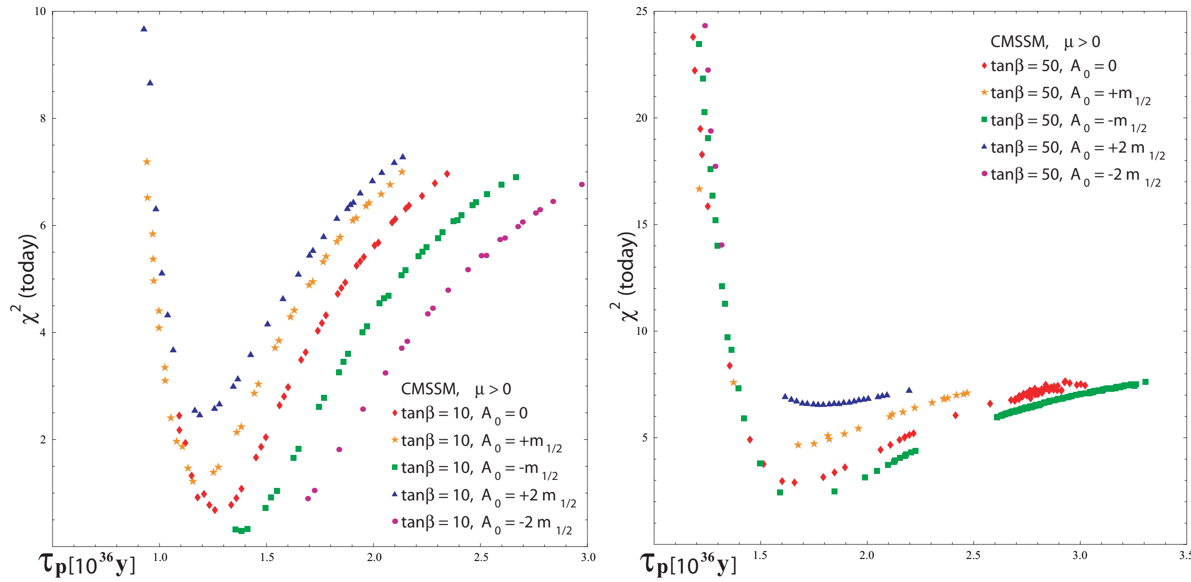

Fig. 4. *A 'χ^2' Best Fit for the Proton Lifetime. Following [52], minimization of the projected error in* $M_W, \sin_e^\theta ff, (g-2)_\mu$ *and* $BR(b \rightarrow s\gamma)$ *is translated into a preferred value of the light threshold correction.*

prising only about 6% of the two-loop effect, it is clear the bulk of the heavy lifting for satisfaction of the $\alpha_s(M_Z)$ matching must be borne by other means. How this task may be distributed between the GUT thresholds and a flipped-type M_{32} scale reduction is the topic of consideration for the next section.

2. The Grand Unified Threshold

As was the case for δ_{light}, we have also here an explicit representation for the corrections induced in the picture of gauge coupling renormalization at the crossing of the heavy GUT mass thresholds in flipped $SU(5)$ [39, 40, 53, 54, 55, 56, 57].

$$\delta_{\text{heavy}} = \frac{\alpha_{em}}{20\pi}\left[-6\ln\frac{M_{32}}{M_{H_3}} - 6\ln\frac{M_{32}}{M_{\bar{H}_3}} + 4\ln\frac{M_{32}}{M_V}\right] \simeq \frac{\alpha_{em}}{20\pi}\left[-6\ln\frac{r^{4/3}g_5^{2/3}}{\lambda_4\lambda_5}\right] \tag{2.27}$$

However, we are not so fortunate as before in terms of our ability to map the available landscape. Whereas the intersection of a highly confining symmetry with an abundance of experimental data gave us some sense of orientation at the light SUSY scale, the Yukawa couplings $\lambda_{4,5}$ are here for example largely unconstrained. Indeed, the absence of H_3 to $\bar{H}_3$ mixing which was essential for protection from fast dimension five proton decay also removes any strong phenomenological constriction from the heavy triplet Higgs supermultiplet masses $M_{H_3} = \lambda_4 |V|$ and $M_{\bar{H}_3} = \lambda_5 |V|$, where $|V|$ is the VEV taken by the $\mathbf{10}$ and $\overline{\mathbf{10}}$ of GUT Higgs. Because of this, flipped $SU(5)$ readily supports $\delta_{\text{heavy}} < 0$ [32] since there is no injunction against $M_{H_3,\bar{H}_3} < M_V = g_5 |V|$ [20]. Curiously, standard $SU(5)$ which must rely exclusively on this mechanism to salvage the lowering of α_s cannot afford such light $(H_3, \bar{H}_3)$, while Flipped $SU(5)$ holds also in abundant reserve the additional proprietary device of a reduced intermediary scale.

We explore in Fig. 5 what the role of the heavy thresholds may be within each of the benchmark [44] models. Without some stronger guiding principles however, the range $-0.0016 < \delta_{\text{heavy}} < 0.0005$ [32] which may be considered plausible is unfortunately quite large. Each labeled strip slides to the left and right based on its prediction for δ_{light}, as centered with the darker narrow (red/blue) bands which reflect the comparatively smaller error in $\sin^2 \theta_W$ from (2.20). Both effects are however significantly overshadowed by uncertainty in the heavy threshold, as demonstrated by the extensive reach of the more lightly shaded (orange/blue) bars. This less well described yet potentially more influential effect threatens to undermine any notion of finely resolved predictability for the $\tau(p \to e^+ \pi^0)$ proton lifetime established to this point. For positive values of δ_{heavy} the $M_{32}^{\max}/M_{32}$ ratio is exaggerated, and most

[20] M_V is the common GUT gauge X, Y and gaugino mass. The parameter $r \equiv \max\{g_5, \lambda_4, \lambda_5\}$ is the largest of the $SU(5)$ and two Yukawa couplings, e.g. taken here as $r = g_5$.

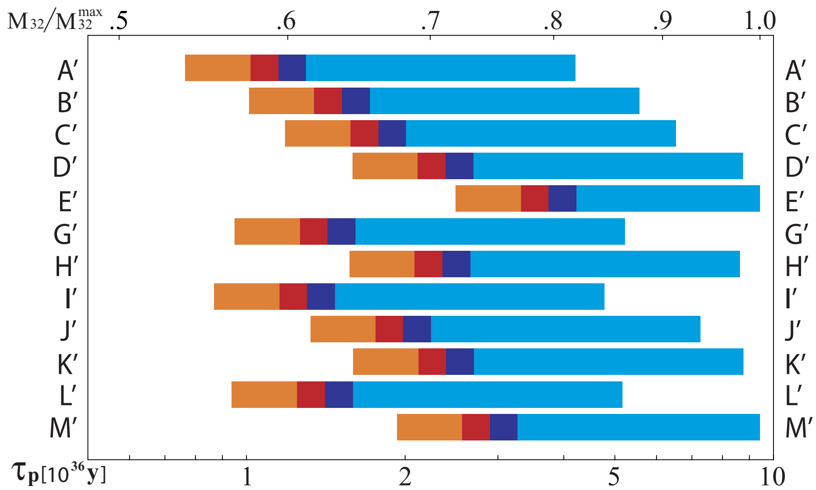

Fig. 5. *Threshold Effects at the Benchmark Points. The overall shifted central prediction for $\tau(p \to e^+\pi^0)$ in each benchmark scenario is determined by the light thresholds. The darkened narrow inner bands (red, blue) represent the experimental uncertainty in $\sin^2\theta_W$, while the lighter outer (orange, blue) strips correspond to plausible variation in the heavy threshold.*

benchmark points become capable of numbers in the low 10^{36} years. However, by the larger freedom allocated to $\delta_{\text{heavy}} < 0$, we cannot rule out $\tau(p \to e/\mu^+\pi^0)$ approaching 10^{37} y in any of these models.

We end this section by looking in closer detail at the relationship between M_{32} and δheavy implied by (2.17,2.21). Selecting now the best light supersymmetric fit from (2.26) and enforcing an experimentally viable (2.20) range for $\alpha_s(M_Z)$, we are provided a strict functional expression isolated in these two variables. This proportional rescaling

$$\frac{M_{32}}{M_{32}^{\text{max}}} \to \left.\frac{M_{32}}{M_{32}^{\text{max}}}\right|_{\delta_{\text{h}=0}} \times e^{-\pi\delta_{\text{h}}/2\alpha_{em}} \tag{2.28}$$

is plotted in the (green) curves of Fig. 6. A strong preference for non-maximal M_{32} is suggested by this figure when restricted to the plausible variations of δ_{heavy} mentioned

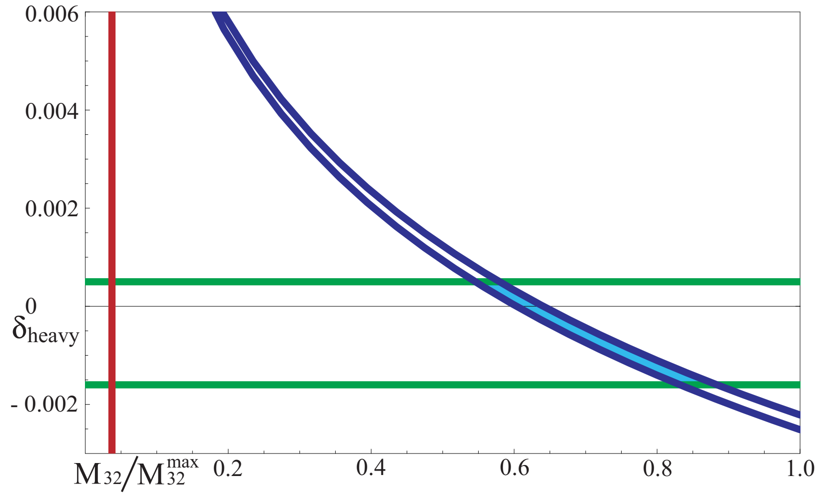

Fig. 6. *Heavy Threshold Effects vs. M_{32}/M_{32}^{max}. The blue curves demonstrate the allowed balance between these factors in flipped $SU(5)$ ($\pm$ the uncertainty in $\sin^2\theta_W$). The green horizontal lines mark the intersection of parameter space with the preferred range of δ_{heavy}, while the red vertical line represents the lower experimental bound on proton decay in the $p \to e^+\pi^0$ mode.*

prior. The cost of our ignorance remains high though, in terms of the expansive real estate still allowed. Indeed it seems a shame to have come so far only to find our fuel expended on the very doorstep of the Planck regime. Perhaps the answer then is not to look behind, but instead to inquire whether the high scale may descend to meet us halfway, and in so bridging that great divide deliver some extra grain of information which could help to constrain our GUT model.

3. Super Unification

It is a recurring theme of our analysis to this point that M_{32} should be shifted downward from its maximal value in conventional $SU(5)$, further separating this 'GUT' group from the fundamental Planck regime $M_{\mathrm{Pl}} = 2 \times 10^{19}$ GeV. However, this is not the whole story, as we are free to consider that (α_1, α_5) further continue their own running upward toward a meeting at some true grand unified scale M_{51}.

$$\frac{1}{\alpha_1} - \frac{1}{\alpha_{51}} = \frac{b_1}{2\pi} \ln \frac{M_{51}}{M_{32}} \tag{2.29}$$

$$\frac{1}{\alpha_5} - \frac{1}{\alpha_{51}} = \frac{b_5}{2\pi} \ln \frac{M_{51}}{M_{32}} \tag{2.30}$$

This process was depicted previously in Fig. 1, where we make special note of the discontinuity in the $U(1)$ progression at M_{32} induced by a remixing with the additional Abelian factor delivered from broken $SU(5)$. A consistent normalization of the known charges within the 'GUT' and SM representations allows us to relate the hypercharge $\alpha_1' \equiv \alpha_Y(M_{32})$ and flipped $SU(5)$ coupling $\alpha_1(M_{32})$ across this boundary.

$$\frac{25}{\alpha_1'} = \frac{1}{\alpha_5} + \frac{24}{\alpha_1} \tag{2.31}$$

The continuance to M_{51} may indeed transport us well beyond M_{32}^{max} and onward closer to M_{Pl}, although the intervening three-plus orders of magnitude remain yet a mighty span. It is of great interest then that string theory generically predicts any lower energy gauge structures emerge intact from a 'super-unification' scale M_{su} that is significantly depressed from the Planck mass. More fascinating yet is the manner in which this reduction is intertwined (in a heterotic context) with the value of the gauge coupling at the super unification:

$$M_{su} = \xi \sqrt{\alpha_{su}}\,, \tag{2.32}$$

where $\xi \lesssim M_{\text{Pl}}$ is a calculable model-dependent parameter [58]. By this assent of the string to make some stretch in our direction, we are tempted to explore the consequence of imposing a full meeting under the identifications:

$$\alpha_{51} \Leftrightarrow \alpha_{su} \quad , \quad M_{51} \Leftrightarrow M_{su} \tag{2.33}$$

It is reasonable first to make a proper counting of parameters unknown either by direct measurement, experimental fit or prediction within a model. The tally of four gauge coupling constants (α_1', α_1, α_5, α_{su}), two mass scales (M_{32}, M_{su}) and the heavy threshold δ_{heavy} leaves us with seven undetermined variables. Counting our available constraints to be the string's mass-coupling relation (2.32), the discontinuity equation at M_{32} (2.31), the two upper renormalization relations in (2.29,2.30), and appropriate combinations of the three lower RGEs (2.12,2.13,2.14), we arrive also at a total of seven. This offers some strong encouragement for the hope of fixing unknowns by such a mechanism.

Our recommended procedure begins with the careful isolation of dependence on δ_{heavy}, as enters via the effective corrections to $\sin^2\theta_W$. We take an inversion of (2.28) for the first of our three choices from the low energy running, evaluated numerically with the aid of (2.17,2.15), and using still the best fit for δ_{light} (2.26).

$$\delta_{\text{heavy}} = \frac{2\alpha_{em}}{\pi} \ln\left(\frac{1.39\times 10^{16}\text{GeV}}{M_{32}}\right) \tag{2.34}$$

For the second, we take the linear combination of (2.12,2.13) which is independent of $\sin^2\theta_W$, eliminating α_1' via (2.31). Finally α_5 is traded out by use of (2.14), which is also satisfactory by itself for the third selection.

$$\alpha_1 = \left[\frac{5}{8\alpha_{em}} - \frac{2}{3\alpha_s} - \frac{(25/24\ b_Y + 5/8\ b_2 - 2/3\ b_3)}{2\pi}\ln\left(\frac{M_{32}}{M_Z}\right)\right]^{-1} \tag{2.35}$$

$$\alpha_5 = \left[\frac{1}{\alpha_s} - \frac{b_3}{2\pi}\ln\left(\frac{M_{32}}{M_Z}\right)\right]^{-1} \tag{2.36}$$

Substituting into (2.29,2.30), we can eliminate both of (α_1, α_5). Using also (2.32) to cut α_{su}, we are left with two expressions in only the variables (M_{32}, M_{su}). The equations are transcendental but may be solved numerically, after which the remaining five unknowns are each only one step removed.

Although this process will always yield some result, we have no guarantee that the output will be real and free from pathologies such as inversion of the fundamental scales. Nevertheless, we may best proceed forward by means of a specific trial. Perhaps naïvely, we will initially choose the missing constants above M_{32} from the context of the heterotic string-derived flipped $SU(5)$ model in [58], while maintaining the existing pure GUT numbers below this scale.

$$b_1 = 43/2 \quad , \quad b_5 = -1 \quad , \quad \xi = \sqrt{4\pi}\, 1.76 \times 10^{18} \text{ GeV} \tag{2.37}$$

Indeed, analysis of these numbers quickly reveals that our reach has exceeded our grasp.

The root of the problem here can be traced to the large size of the constant b_1, which causes α_1 to climb much more rapidly than did the hypercharge. Because of this, M_{51} actually remains always less than M_{32}^{max}, never approaching the high energies which were sought. Recalling that by definition M_{51} converges exactly to M_{32}^{max} in the case of triple unification ($\alpha_1 = \alpha_5$), we should though be able to parametrically distinguish in which (if any) cases it splits instead to a higher mass as M_{32} is lowered. This is indeed accomplished by differentiation of the following expression obtained from (2.29,2.30) and supplemented with (2.35,2.36).

$$M_{51} = M_{32} \times \exp\left\{ \frac{2\pi}{b_1 - b_5} \left(\frac{1}{\alpha_1} - \frac{1}{\alpha_5} \right) \right\} \tag{2.38}$$

The stipulation that M_{51} vary conversely to M_{32} is then enforced in terms of the five beta function coefficients.

$$\frac{dM_{52}}{dM_{32}} < 0 \quad \Longrightarrow \quad \frac{25/24\ b_Y + 5/8\ b_2 - 5/3\ b_3}{b_1 - b_5} > 1 \tag{2.39}$$

Evaluating the CMSSM numerically for the conventional scenario of b_1 positive (ascending) and b_5 negative, this reduces to:

$$(b_1 - b_5) < 12.5 \tag{2.40}$$

The condition is clearly violated by the constants from (2.37). However even if met, this extremal transition is still not alone sufficient to enact the desired super unification.

Once (2.40) is satisfied, M_{51} will indeed climb upward, but we are restricted by how far it may be pushed in terms of how low we are willing to let M_{32} descend. Proton decay limits are not terribly strong here as has been mentioned, allowing $M_{32}/M_{32}^{\max} \sim .1$ before difficulties are encountered. However, this pushes the heavy threshold to $\sim +.008$, far exceeding the region formerly considered most plausible. Taking instead that upper bound of $\delta_{\text{heavy}} \lesssim .0005$ is much more confining, as suggested by Fig. (6), corresponding to $M_{32}/M_{32}^{\max}$ above approximately one-half. We shall be a bit more generous in what follows, allowing this ratio to drop to around one-quarter. Also, the ostensible super-unified coupling

$$\alpha_{51} = \left[\frac{1}{(1 - b_5/b_1)\,\alpha_5} + \frac{1}{(1 - b_1/b_5)\,\alpha_1} \right]^{-1} , \tag{2.41}$$

is found to be quite stable in against such variation and may be safely set near the standard value 1/24. We can then establish some heuristic limits on when M_{51} from (2.38) is reasonably capable of an extension to M_{su} as defined in (2.32).

$$(b_1 - b_5) \;\lesssim\; \frac{17}{6.6 - \ln\,(M_{\text{Pl}}/\xi)} \tag{2.42}$$

Taking ξ from (2.37) provides a much tighter constraint than (2.40), demanding the beta function difference drop almost to 3. By way of reference for $b_1 = 2$, the heavy threshold is here $+.004$, and the proton lifetime ($p \to e^+\pi^0$) is a very youngish 5.4×10^{34} y. We note this choice might more easily pass the casual observer unnoted than the conspicuously large suggestion of 43/2. Nevertheless, it appears by general arguments that such a value is quite unattainable [59, 60, 61], simply because negative beta function contributions as would be required to help cancel the large number of postively contributing loops arise only from non-Abelian self-interactions, of which the group $U(1)$ has, by definition, none.

It is certainly possible however to get a little aid from the parameter ξ, although the difference limit moves for example only marginally above five when ξ is cut by a full order of ten. A potential remixing of couplings at M_{su}, and additional loop and threshold effects may all play at least some small role as the mechanism for shifting these parameters toward a more favorable range. This is in addition to the need for more consistent application of a single model across the M_{32} transition, and without any general survey of what competing string models might predict. It is possible though that we have been too cautious in anticipation of some movement by the heavy scale. The presence of some (slightly) enlarged extra dimension [62] may indeed be capable of bringing the Super-Unification well down into the $10^{(15-16)}$ GeV range, preserving the heart of this picture, even as M_{SU} descends below the traditional value of $M_{32}^{\max}$. As with many good things however, we suggest that this mechanism is best when applied in some moderation.

It has been natural in the context of flipped $SU(5)$ to expect that the split couplings continue upward to a true unification. It is in fact only in this way that we can hope to recover the beneficial GUT properties such as a correlated charge quantization and successful prediction of m_b/m_τ [21]. Such memories of a simpler past are naturally inherited if $SU(5)\times U(1)$ is descendant from a structure such as $SO(10)$

[21] Although divergence from this relation for the lighter generations is sometimes mentioned as a failure of Grand Unification, we take the view that competing $\mathcal{O}$(MeV) corrections must in fact be expected to wash out masses in that same order.

[22]. That the string is willing to provide suitable representations for this purpose in vicinity of the necessarily escalated GUT scale is quite a strong enticement for the union of these pictures. Despite incompatibility of the example model, we have shown that a modest softening of b_1 relative to b_Y is sufficient to bring this super unification into full fruition. Furthermore, there are generic and novel predictions which must hold if this scenario is to play out. Namely, δ_{heavy} is expected to work in the same direction as δ_{2loop}, which extends the deficit from which the flipped scale reduction must rescue $\alpha_s(M_Z)$ and leads in turn to much more rapid proton decay. While recognizing the dangers of building speculation too high upon itself, it is fascinating to consider that this analysis might be turned around to in fact constrain the selection of a string derived model by merit of the beta function (b_1, b_5) and scale factor ξ predictions which it makes.

4. String Theory Considerations

It appears that the superstring framework is ideally suited to match our expectations from 'low-energy' physics, bearing with simplicity, consistency and grace the origins of covariant locally gauged chiral GUT matter multiplets in replicated families. The appeal of this proposal is amplified moreover by the unified intrinsic inclusion of both space-time supersymmetry and a gravitational sector (local SUSY). Although the focus here has been to allow the more tangible GUT content to steer our choices among the myriad possibilities allowed in string model building, it is only fair in turn to allow string theory some suggestion of what GUT constructions it may prefer. For the heterotic string, it is not possible to find adjoint group representations at the first

[22]It is of great interest that the **16** spinor of $S0(10)$ breaks to the quantum numbers of only the flipped variety $SU(5)$, while the $(\mathbf{5}, \bar{\mathbf{5}})$ of SM Higgs with conventional charges readily emerge from a fundamental **10**.

level Kac-Moody algebra. This has historically been considered an additional selling point for flipped $SU(5)$, which characteristically employs GUT Higgs representations $(\mathbf{10}, \bar{\mathbf{10}})$ similar to those used for matter content. Moreover, it is common for the $SU(5) \times U(1)_x$ to descend from an enclosing factor of $SO(10)$, under which the spinor $\mathbf{16}$ representation decomposes neatly as a full matter generation $(\mathbf{10} \oplus \bar{\mathbf{5}} \oplus \mathbf{1})$, with quantum numbers in the $\bar{\mathbf{5}}$ appropriate to a flipped-type assignation. Conversely, the EW Higgs $(\mathbf{5}, \bar{\mathbf{5}})$ carry the charges of standard $SU(5)$, and naturally break from a fundamental $\mathbf{10}$. In the newer interesecting D-brane approach, it has been fascinating to observe a parallel situation. Since the five-plets present in this scenario arise bi-fundamentally from the intersection of a single D-brane with a second set of five branes, and since there are two distinct pairings (plus conjugates) related to mixed application of the orientifold mirror, the net $U(1)$ charges of such a state may be composed either by the sum or by the difference of charges from the two stacks. Again, this distinction which necessarily arises from the construction is a perfect match to the two variety of five-plets in flipped $SU(5)$.

The other side of this coin, is that if we are going to imagine that our preferred GUT group descends out of a string theoretical construction near the Planck scale, then it is also incumbent upon us to inquire what string theory would directly add to or remove from the predictions of a GUT not so derived. We defer here to the analysis of [63], which considers proton lifetime within a pure intersecting D-brane context, and make the necessary conversions into a flipped-$SU(5)$ language. The most interesting result, is that the multiplicative prefactor which emerges after many laborious calculations is simply of order unity. In other words, it appears safe to consider at least this question of proton lifetime in the basic GUT scenario. However, the same work makes another suggestion which may have strange relevance to the flipped picture. It seems that D-brane constructions may systematically block dimension-six decay via the $\bar{\mathbf{5}}$ operator channel. Recalling that this is the only channel which flipped $SU(5)$ preserves, the joke may well be on us. If future searches for proton decay continue to return null results well past the 10^{36} year range, this conspiracy between the string and flipped-GUT may need to be revisited as a possible culprit.

D. Conclusions

Proton decay has been an essential piece long missing from the Grand Unified Theoretical puzzle. As with any puzzle, the contours and texture of the pieces already in place give us much information regarding those still lacking. And conversely, the shape of the piece in hand also tells us much about the location in which it may be made to reside.

It has been pointed out that the flipped variety $SU(5)$ GUT avoids catastrophically rapid dimension five proton decay while successfully matching the low energy strong coupling, naturally accommodates the now essential right handed neutrino, provides intrinsically for doublet-triplet Higgs splitting, and may more effortlessly conform to a string theoretical embedding. We have seen the essential connectedness of supersymmetry to Grand Unification, which by a single step toward the Planck scale dramatically improves actual convergence of the SM couplings, and also salvages dimension six proton decay from current experimental limits. Moreover, the bounty of phenomenological constraint on the CMSSM has been sufficient to isolate a strongly preferred region of parameter space whose influences are transmitted to our GUT construction in guise of corrections at the the light SUSY mass thresholds. The novel flipped splitting of fundamental scales allows for still safe, yet more imminently observable dimension six decay while also pointing toward the prospect of a true super unification delayed to include gravity on its own terms and territory at the reduced string scale. While not so favored by a wealth of direct corollary evidence, we have also established the significant role which must be played by the heavy GUT scale thresholds (and potentially string-derived constants of the running) if this final margin is to be traversed.

We look now with anticipation toward the next generation of massive water-Čerenkov detectors, knowing that each piece of the puzzle newly added presents its own fresh perimeter for the next matching, and hastens the day on which we shall take in the entire vista as a whole with unobstructed sight.

CHAPTER III

HETEROTIC STRING MODEL BUILDING

A. Minimal Superstring Standard Models (MSSMs)

The approach we favor here for model production uses free fermions for the internal degrees of freedom of a heterotic string theory,[64, 65] as in the "FNY" model of ref. [66, 67] with its $SU(3)_C \times SU(2)_L \times U(1)_Y \times \prod_i U(1)_i$ observable gauge group. Preservation of supersymmetry mandates the acquisition of non-zero vacuum expectation values (VEVs) to cancel the Fayet-Iliopoulos (FI) D-term, which arises in conjunction with the anomalous $U(1)$ factor endemic to many related constructions, while keeping all other D- and F-terms zero as well. In this context, the assignment of a satisfactory ground state is a delicate and confining business, but some freedom does remain to tailor phenomenology in the emerging low energy effective field theory. Success has been had here with VEVs decoupling all Standard Model charged fields outside the MSSM [68, 69, 15, 16], and mechanisms of generational mass suppression have arisen from powers of $\frac{\langle \phi \rangle}{M_P}$ with non-renormalizable terms and from coupling to Higgs fields with differing contributions to the massless physical combination. Promising, if imperfect, particle properties have been realized and discussed, but indications exist from varied directions [16, 17, 70, 71, 72, 73, 74] that attempts restricted to a non-Abelian singlet vacuum are unsatisfactory, and that perhaps nature's craft avails a larger set of tools. This paper will then focus generally on the technology of assigning VEVs to non-Abelian fields, and in particular on the geometrical framework that is introduced by the presence of VEV components within a group space. The geometrical point of view facilitates manipulations which emerge for non-Abelian VEVs, such as treating superpotential contractions with multiple pairings, and examining the new possibility of self-cancellation between elements of a single term. When expressed in this language, the process of describing valid solutions, or compatibilities between the F and D conditions, can become more accessible and intuitive, and will hopefully aid in closing the gap between string model building and low energy experimental evidence.

The sequence taken will be first to review the constraints which supersymmetry imposes on our VEV choices, in particular for the non-Abelian case. Next, the discussion will be made concrete by turning to the case of $SU(2)$, and following that, $SO(2n)$. These choices are made because of their application to the string-derived FNY MSSM, and flipped $SU(5)$ GUT [75, 76], via $SO(6)$, respectively, but other benefits exist as well. $SU(2)$ is well known from the theory of spin-$\frac{1}{2}$ systems, and as a rank 1 group with a number of generators equal to its fundamental dimension (3), it represents the simplest specific case on which to initiate discussion. $SO(2n)$, on the other hand, will generally represent one of the four main Lie group classifications, and introduces new complications by way of higher rank groups and an adjoint space of dimension greater than the fundamental. Furthermore, although the fields under consideration are all space-time scalars, superpotential terms can inherit an induced symmetry property from the analytic rotationally invariant contraction form of the group under study. $SU(2)$ will have a "fermionic" nature, with an antisymmetric contraction, while that of $SO(2n)$ will be symmetric, or "bosonic". Interspersed in the document body will be special topics, such as self-cancellation, and specific examples of superpotential terms. The final section will be concluding remarks.

B. D- and F-Flatness Constraints

The well known requirements for preservation of space-time supersymmetry, as expressed in the so-called F and D-terms have been reviewed in [68, 69, 15, 16, 17]. They will again be summarized here,[1], with a new emphasis on geometric interpretation of the non-Abelian VEVs.

Space-time supersymmetry is broken in a model when the expectation value of the scalar potential,

$$V(\varphi) = \frac{1}{2}\sum_{\alpha} g_{\alpha}^{2}\left(\sum_{a=1}^{\dim(\mathcal{G}_\alpha)} D_a^{\alpha} D_a^{\alpha}\right) + \sum_i |F_{\varphi_i}|^2 \ , \tag{3.1}$$

becomes non-zero. The D-term contributions in (3.1) have the form,

$$D_a^{\alpha} \ \equiv \ \sum_m \varphi_m^{\dagger} T_a^{\alpha} \varphi_m \ , \tag{3.2}$$

with T_a^{α} a matrix generator of the gauge group $\mathcal{G}_\alpha$ for the representation φ_m. The F-term contributions are,

$$F_{\Phi_m} \ \equiv \ \frac{\partial W}{\partial \Phi_m} \ . \tag{3.3}$$

The φ_m are (space-time) scalar superpartners of the chiral spin-$\frac{1}{2}$ fermions ψ_m, which together form a superfield Φ_m. Since all of the D and F contributions to (3.1) are positive semidefinite, each must have a zero expectation value for supersymmetry to remain unbroken.

For an Abelian gauge group, the D-term (3.2) simplifies to

$$D^{i} \ \equiv \ \sum_m Q_m^{(i)} |\varphi_m|^2 \tag{3.4}$$

[1] Portions of this section, most notably regarding flatness constraints and the group $SU(2)$, have been previously reported in ref. [17].

where $Q_m^{(i)}$ is the $U(1)_i$ charge of φ_m. When an Abelian symmetry is anomalous, that is, the trace of its charge over the massless fields is non-zero,

$$\mathrm{Tr}\, Q^{(A)} \neq 0 \ , \tag{3.5}$$

the associated D-term acquires a Fayet-Iliopoulos (FI) term, $\epsilon \equiv \frac{g_s^2 M_P^2}{192\pi^2} \mathrm{Tr}\, Q^{(A)}$,

$$D^{(A)} \equiv \sum_m Q_m^{(A)} |\varphi_m|^2 + \epsilon \, . \tag{3.6}$$

g_s is the string coupling and M_P is the reduced Planck mass, $M_P \equiv M_{Planck}/\sqrt{8\pi} \approx 2.4 \times 10^{18}$ GeV. It is always possible to place the total anomaly into a single $U(1)$.

The FI term breaks supersymmetry near the string scale,

$$V \sim g_s^2 \epsilon^2 \ , \tag{3.7}$$

unless it can be canceled by a set of scalar VEVs, $\{\langle \varphi_{m'} \rangle\}$, carrying anomalous charges $Q_{m'}^{(A)}$,

$$\langle D^{(A)} \rangle = \sum_{m'} Q_{m'}^{(A)} |\langle \varphi_{m'} \rangle|^2 + \epsilon = 0 \ . \tag{3.8}$$

To maintain supersymmetry, a set of anomaly-canceling VEVs must simultaneously be D-flat for all additional Abelian and the non-Abelian gauge groups,

$$\langle D^{i,\alpha} \rangle = 0 \ . \tag{3.9}$$

A consistent solution to all (3.9) constraints specifies the overall VEV "FI-scale", $\langle \alpha \rangle$, of the model. A typical FNY value is $\langle \alpha \rangle \approx 7 \times 10^{16}$ GeV.

C. The group $SU(2)$

1. The $SU(2)/SO(3)$ Connection

For the case of $SU(2)$, $T_a^{SU(2)}$ will take on the values of the three Pauli matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.10}$$

Each component of the vector $\vec{D}$ in this internal space will be the total, summed over all fields of the gauge group, "spin expectation value" in the given direction. Vanishing of the $\langle \vec{D} \cdot \vec{D} \rangle$ contribution to $\langle V \rangle$ demands that $SU(2)$ VEVs be chosen such that the total $\hat{x}, \hat{y}$, and $\hat{z}$ expectation values are individually zero. The normalization length, $S^\dagger S$, of a "spinor" S will generally be restricted to integer units by Abelian D-flatness constraints from the Cartan sub-algebra and any extra $U(1)$ charges carried by the doublet (cf. Eq. 3.4 with $S^\dagger S$ playing the role of $|\varphi|^2$). Each spinor then has a length and direction associated with it and D-flatness requires the sum, placed tip-to-tail, to be zero. This reflects the generic non-Abelian D-flatness requirement that the norms of non-Abelian field VEVs are in a one-to-one association with a ratio of powers of a corresponding non-Abelian gauge invariant [77].

It will be useful to have an explicit (normalized to 1) representation for $S(\theta, \phi)$. This may be readily obtained by use of the rotation matrix,

$$R(\vec{\theta}) \equiv e^{-i\frac{\vec{\theta}\cdot\vec{\sigma}}{2}} = \cos(\frac{\theta}{2}) - i\hat{\theta} \cdot \vec{\sigma} \sin(\frac{\theta}{2}) \ , \tag{3.11}$$

to turn $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv | + \hat{z}\rangle$ through an angle θ about the axis $\hat{\theta} = -\hat{x}\sin\phi + \hat{y}\cos\phi$.

The result, $\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{i\phi} \end{pmatrix}$, is only determined up to a phase and the choice

$$S(\theta,\phi) \equiv \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}\, e^{+i\frac{\phi}{2}} \end{pmatrix} \tag{3.12}$$

will prove more convenient in what follows. Within the range of physical angles, $\theta = 0 \to \pi$ and $\phi = 0 \to 2\pi$, each spinor configuration is unique (excepting ϕ phase freedom for $\theta = 0, \pi$) and carries a one-to-one geometrical correspondence. Up to a complex coefficient, the most general possible doublet is represented.

A non-trivial superpotential W additionally imposes numerous constraints on allowed sets of anomaly-canceling VEVs, through the F-terms in (3.1). F-flatness (and thereby supersymmetry) can be broken through an n^{th}-order W term containing Φ_m when all of the additional fields in the term acquire VEVs,

$$\langle F_{\Phi_m} \rangle \;\; \sim \;\; \langle \frac{\partial W}{\partial \Phi_m} \rangle \sim \lambda_n \langle \varphi \rangle^2 (\frac{\langle \varphi \rangle}{M_{str}})^{n-3} \;, \tag{3.13}$$

where φ denotes a generic scalar VEV. If Φ_m also carries a VEV, then supersymmetry can be broken simply by $\langle W \rangle \neq 0$. For both practical and philosophical reasons, the (Abelian) D-condition is usually enforced first. Unless this constraint holds, supersymmetry will be broken near M_P. On the other hand, the F-condition is not all-or-nothing, since the *order* of a given dangerous term fixes the scale at which SUSY fails[2]. F-flatness must be retained up to an order in the superpotential that is consistent with observable sector supersymmetry being maintained down to near the electroweak (EW) scale. However, it may in fact be *desirable* to allow such a term to escape at some elevated order, since it is known that supersymmetry does *not* survive

[2] The lower the order of an F-breaking term, the closer the supersymmetry breaking scale is to the string scale.

down to ‘everyday’ energies. Depending on the string coupling strength, F-flatness cannot be broken by terms below eighteenth to twentieth order[3].

2. Non-Abelian Flat Directions and Self-Cancellation

In [69] we classified MSSM producing singlet field flat directions of the FNY model and in [15] we studied the phenomenological features of these singlet directions. Our past investigations suggested that for several phenomenological reasons, including production of viable three generation quark and lepton mass matrices and Higgs h-$\bar{h}$ mixing, non-Abelian fields must also acquire FI-scale VEVs.

In our prior investigations we generally demanded “stringent” flatness. That is, we forced each superpotential term to satisfy F-flatness by assigning no VEV to at least two of the constituent fields. While the absence of any non-zero terms from within $\langle F_{\Phi_m} \rangle$ and $\langle W \rangle$ is clearly sufficient to guarantee F-flatness along a given D-flat direction, such stringent demands are not necessary. Total absence of these terms can be relaxed, so long as they appear in collections which cancel among themselves in each $\langle F_{\Phi_m} \rangle$ and in $\langle W \rangle$. It is desirable to examine the mechanisms of such cancellations as they can allow additional flexibility for the tailoring of phenomenologically viable particle properties while leaving SUSY inviolate.[4] It should be noted that success along these lines may be short-lived, with flatness retained in a given order only to be lost at one slightly higher.

Since Abelian D-flatness constraints limit only VEV magnitudes, we are left with the gauge freedom of each group (phase freedom, in particular, is ubiquitous) with which to attempt a cancellation between terms (whilst retaining consistency with

[3] As coupling strength increases, so does the required order of flatness.

[4] Research along this line for the FNY MSSM is currently underway.

non-Abelian D-flatness). However, it can often be the case that only a single term from W becomes an offender in a given $\langle F_{\Phi_m}\rangle$ (cf. Table 1B of [17]). If a contraction of non-Abelian fields (bearing multiple field components) is present it may be possible to effect a *self-cancellation* that is still, in some sense, "stringently" flat.

Near the string scale the complete FNY gauge group is

$$\begin{aligned}&[SU(3)_C \times SU(2)_L \times U(1)_C \times U(1)_L \times U(1)_A \times \prod_{i=1'}^{5'} U(1)_i \times U(1)_4]_{\rm obs} \times \\ &[SU(3)_H \times SU(2)_H \times SU(2)_{H'} \times U(1)_H \times U(1)_7 \times U(1)_9]_{\rm hid}\,. \end{aligned} \tag{3.14}$$

The FNY non-Abelian hidden sector fields are triplets of $SU(3)_H$ or doublets of $SU(2)_H$ or $SU(2)_{H'}$. Self-cancellation of F-terms, that would otherwise break observable sector supersymmetry far above the electro-weak scale, might be possible for flat directions containing such doublet or triplet VEVs. Since intermediate scale $SU(3)_H$ triplet/anti-triplet condensates are more likely to produce viable observable sector electro-weak scale supersymmetry breaking than are their $SU(2)_{H^{(\prime)}}$ counterparts, we focus herein on non-Abelian directions containing doublet, but not triplet, FI-scale VEVs.

Whenever "spinors" of $SU(2)$ appear in W, they are not of the form $S^{\dagger}S$, but rather are in the antisymmetric contraction

$$S_1 \cdot S_2 \equiv S_1^T\, i\sigma_2\, S_2 = S_1^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} S_2 \;. \tag{3.15}$$

This form, which avoids complex conjugation and thus satisfies the requirement of analyticity, is also rotationally (gauge) invariant as can be verified using $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$, $[\sigma_2, \sigma_2] = 0$, and Eqs. (3.10, 3.11) :

$$\sigma_2 R(\vec{\theta}) = R^*(\vec{\theta})\sigma_2 \tag{3.16}$$

$$S_1' \cdot S_2' = (RS_1)^T(i\sigma_2)(RS_2) = S_1^T(i\sigma_2)(R^\dagger R)S_2 = S_1 \cdot S_2\,. \tag{3.17}$$

From Eq. (3.12), the general form of such a contraction may be written explicitly as

$$S(\theta,\phi)\cdot S(\Theta,\Phi) = -\sin(\frac{\theta-\Theta}{2})\cos(\frac{\phi-\Phi}{2}) - i\sin(\frac{\theta+\Theta}{2})\sin(\frac{\phi-\Phi}{2})\ . \tag{3.18}$$

The magnitude of this term must be a purely geometrical quantity and can be calculated as

$$|S(\hat{n})\cdot S(\hat{N})| = \sqrt{\frac{1-\hat{n}\cdot\hat{N}}{2}} = \sin(\frac{\delta}{2})\ , \tag{3.19}$$

where $\delta(0 \to \pi)$ is the angle between $\hat{n}$ and $\hat{N}$. The absence of a similar concise form for the phase is not a failing of rotational invariance, but merely an artifact of the freedom we had in choosing (3.12). Self-cancellation of this term is independent of the spinors' lengths and demands only that their spatial orientations be parallel[5]. The same conclusion is reached by noting that anti-symmetrizing the equivalent (or proportional) spinors yields a null value. VEVs satisfying this condition are clearly not D-consistent unless other non-Abelian VEVed fields also exist such that the *total* "spin" vector sum remains zero[1]. To examine generic cases of cancellation between multiple terms, the full form of (3.18) is needed.

As an important special case, consider the example of a superpotential term $\phi_1 \ldots \phi_n S_1 S_2 S_3 S_4$[2] with ϕ_n Abelian. This is shorthand for an expansion in the various pairings of non-Abelian fields,

$$W \propto \phi_1 \ldots \phi_n \{(S_1\cdot S_2)(S_3\cdot S_4) + (S_2\cdot S_3)(S_1\cdot S_4) + (S_3\cdot S_1)(S_2\cdot S_4)\}\ , \tag{3.20}$$

[5]The contraction of a field with itself vanishes trivially.

[1]In the notation of [15], taking a single sign for each of the $s_{k'}$ is a special case of non-Abelian self-cancellation, as is $\sum_{k=1}^{p} n_k s_k = 0$ a special case of the D-constraint.

[2]Here and in the following discussion we consider the doublets of a single symmetry group.

Broadly, we notice that:

- Whenever each term holds the same field set, the spinors may be treated as normalized to one, bringing any larger magnitudes out front as overall factors. Furthermore, since S^T appears but never $S^\dagger$, the same can be done with any phase selections.

- Since the contractions are antisymmetric, sensible interpretation of terms with multiple factors demands the specification of an ordering.

The appropriate ordering, or equivalently the choice of relative signs, for (3.20) is such to ensure *total* anti-symmetrization. When (3.20) is explicitly evaluated using the previously established formalism it is seen to vanish identically for *all* field values. The calculation is simplified without loss of generality by taking $\theta_1 = \phi_1 = \phi_2 = 0$. We emphasize the distinction between this identical exclusion from the superpotential and cancellations which exist only at the vacuum expectation level. W-terms with 6 non-Abelian fields are formed with factors of (3.20) and also vanish, as do all higher order terms.

Even safe sectors of W (in particular with $\langle \Phi_m \rangle = 0$) may yield dangerous $\langle F_{\Phi_m} \rangle \equiv \langle \frac{\partial W}{\partial \Phi_m} \rangle$ contributions. The individual F-terms may be separated into two classes based on whether or not Φ_m is Abelian. For the case of Φ_m non-Abelian, $\langle F_{\Phi_m} \rangle$ is itself a doublet. As a note, terms like $\langle F_{S_4} \rangle \equiv \langle \frac{\partial W}{\partial S_4} \rangle$ which *would have* arisen out of (3.20) are cyclically ordered and also vanish identically.

D. Minimal Standard Heterotic-String Model Non-Abelian Flat Directions

Our initial systematic search for MSSM-producing stringent flat directions revealed four singlet directions that were flat to all order, one singlet direction flat to twelfth order, and numerous singlet directions flat only to seventh order or lower [69]. For these directions, renormalizable mass terms appeared for one complete set of up-, down-, and electron-like fields and their conjugates. However, the apparent top and bottom quarks did not appear in the same $SU(2)_L$ doublet. Effectively, these flat directions gave the strange quark a heavier mass than the bottom quark. This inverted mass effect was a result of the field Φ_{12} receiving a VEV in all of the above directions.

We thus performed a search for MSSM-producing singlet flat directions that did not contain $\langle\Phi_{12}\rangle$. None were found. This, in and of itself, suggests the need for non-Abelian VEVs in more phenomenologically appealing flat directions. Too few first and second generation down and electron mass terms implied similarly.

Among the FNY MSSM non-Abelian flat directions investigated in [16, 17], that denoted FDNA(5+8) generated the best phenomenology. The search for dangerous terms yielded 131 results, five of them to $\langle W\rangle$ with 11^{th} the lowest order and 126 of them to $\langle F\rangle$, as low as order four (counting variations of the four fields labeled (Φ_4) only once). World sheet selection rules reduced this number to 32, all of them F-terms. Disallowing more than two non-Abelian fields (for each $SU(2)$ group) trimmed the list further to just the eight terms in Table I. If a single incidence of (Φ_4) is mandated, then it is so indicated by a lack of parenthesis.

The lowest order potentially dangerous F-term (designated as #1) contains a factor of $\langle H_{26}\cdot\mathbf{V}37\rangle$ which we would like to cancel, as per the discussion of Section 2. This requires the VEV orientations to be chosen parallel in the three-dimensional $SU(2)_H$ adjoint space. Since FDNA(5+8) contains (two) additional non-Abelian fields with VEVs ($\mathbf{V}5$ and $\mathbf{V}35$) which can oppose H_{26} and $\mathbf{V}37$ with an equal total magnitude, this choice is also D-consistent. The same factor appears in and eliminates additionally dangerous Table I F-terms #2,5,7 and 8. Since the other two non-Abelian VEVs had to be parallel as well, the contraction $\langle\mathbf{V}5\cdot\mathbf{V}35\rangle$ in term #6 is also zero.

Table I. Surviving Candidates for non-Abelian Cancellation

#	O(W)	F-term
1	4	$H^s_{16}\,\langle H_{26}\cdot\mathbf{V}37\rangle\langle N^c_3\rangle$
2	5	$V^s_{32}\,\langle H_{26}\cdot\mathbf{V}37\rangle\langle\Phi_4 H^s_{37}\rangle$
3	5	$V_{15}\,\langle\cdot\mathbf{V}35\rangle\langle\Phi^{\prime}_4 H^s_{30}H^s_{21}\rangle$
4	5	$V_{17}\,\langle\cdot\mathbf{V}5\rangle\langle\Phi^{\prime}_4 H^s_{30}H^s_{15}\rangle$
5	8	$\overline{\Phi}_{13}\,\langle H_{26}\cdot\mathbf{V}37\rangle\langle(\Phi_4)H^s_{31}H^s_{30}H^s_{15}N^c_3\rangle$
6	9	$\overline{\Phi}_{13}\,\langle\mathbf{V}5\cdot\mathbf{V}35\rangle\langle\Phi_{23}(\Phi_4)^2 {H^s_{30}}^2 H^s_{21}H^s_{15}\rangle$
7	9	$\overline{\Phi}_{12}\,\langle H_{26}\cdot\mathbf{V}37\rangle\langle\Phi_{23}(\Phi_4)H^s_{31}H^s_{30}H^s_{15}N^c_3\rangle$
8	10	$H^s_{36}\,\langle H_{26}\cdot\mathbf{V}37\rangle\langle\Phi_{23}\Phi_4 H^s_{31}H^s_{30}H^s_{15}H^s_{37}N^c_3\rangle$

In the language of [16], we could have said $s_{H_{26}} = s_{\mathbf{V}37} = 1$ and $s_{\mathbf{V}5} = s_{\mathbf{V}35} = -1$. This leaves us with only #3 and #4, both of which are fifth order terms with un-VEVed *non-Abelian* fields so that self-cancellation is impossible. Furthermore, they will appear in different F-terms and each allows only a single (Φ_4) configuration, ruling out a couple of other (less satisfactory) scenarios. The choice $\langle\Phi^{\prime}_4\rangle = 0$, along with $\langle\overline{\Phi}^{\prime}_4\rangle = 0$ for consistency with Eqs. (3.21,3.22) of [17], would restore F-flatness by simply removing the offending terms from $\langle F\rangle$. However, as was discussed in [17], this seems phenomenologically inviable and so it appears that we are stuck with a broken FDNA(5+8) at order five.[3] Also, while it is common to see the vanishing of terms with excessive non-Abelian doublets, these mark the *only* examples wherein non-Abelian self-cancellation by selected VEVs was found for the 'Table 1A' (of [17])

[3]As a note, the cancellations which were successful are insensitive to the factor of 18 between flat directions FDNA5 and FDNA8. (See Table 1.A of [16].)

flat directions.

E. The $SO(2n)$ Lie Groups

1. General Properties

We will now turn our attention to the general case of $SO(2n)$, the rotation group in an even dimensional space. Wherever a concrete example is needed, $SO(6)$ will be focused on for the sake of its relevance to existing studies [25, 78, 79, 80, 81, 70, 82, 83, 84, 85] of a string-derived flipped $SU(5)$ GUT. It is seen quickly in these cases that certain coincidences and simplicities afforded by the $SU(2)/SO(3)$ example are no longer available, and that our techniques have to be adjusted accordingly. Firstly, the case of rotation in three dimensions is very special. Any transformation is facilitated by the use of three angles and three generators. There is a unique axis normal to to any rotation plane and our three rotation generators, which define the adjoint space, can be labeled in one-to-one correspondence with unit vectors from the fundamental space. It is only in this case that the familiar cross-product can be defined. Secondly, since all generators fail to commute, we have in $SU(2)$ the simplest case of a rank 1 group, with only a single diagonalizable matrix.

Since the elements of a transformation between coordinate sets are the projections of unit rows from one basis onto unit columns of another, the transpose of this matrix must interchange these roles and yield the inverse operator. This property, named orthogonality for the nature of the eigenvectors used to diagonalize symmetric matrices, is the origin of the familiar "dot product" rotational invariant. Orthogonal groups are spanned by antisymmetric generators,

$$\left(e^{M}\right)^{T} = e^{M^{T}} = e^{-M}\,, \tag{3.21}$$

and by satisfying the condition that

$$\det(e^{M}) = e^{\mathrm{Tr}\,(M)} = e^{0} = 1\,, \tag{3.22}$$

are also licensed to bear the designation of "special". The $(i,j)^{\mathrm{th}}$ element of the rotation generator between Cartesian axes $\hat{a}$ and $\hat{b}$ is determined in the small angle limit:

$$\left(M^{ab}\right)_{ij} = \delta_i^a \delta_j^b - \delta_i^b \delta_j^a\,. \tag{3.23}$$

As with all rotation operators, these matrices respect the algebra

$$[M^{ab}, M^{cd}] = \delta^{ad} M^{bc} + \delta^{bc} M^{ad} - \delta^{ac} M^{bd} - \delta^{bd} M^{ac}\,, \tag{3.24}$$

which is specified almost entirely by group closure and symmetries. We will prefer to multiply each group element by a factor of i and go to a trivially complex Hermitian representation.

The number of generators required in an m dimensional space is the number of ways to take pairs of axes, $\binom{m}{2} \equiv \frac{m!}{2!(m-2)!} = \frac{m(m-1)}{2}$, or equivalently, the number of elements composing a general antisymmetric $m \times m$ matrix. For the case of $SO(6)$, there are 15 possible rotation planes, and 15 corresponding angles needed to specify the general rotation. Although only five angles are needed to provide the orientation of a 6-vector, there remains an orthogonal 5-space, with four angles, which leaves the vector intact. Perpendicular to this rotation is a 4-space with three angles, and so emerges another way of counting the $\sum_{\alpha=1}^{m-1} \alpha = \frac{m(m-1)}{2}$ generators needed to enact all possible transformations.

Certain consequences can be seen immediately when the number of generators exceeds the fundamental group dimension. First, the conception of a rotation axle must be abandoned in favor of the rotation plane. Secondly, we see that spanning the adjoint space will require an even *larger* basis, and we cannot expect that transformations in the fundamental space will be able to realize generic 'expectation value' orientations in the chosen generator set. Since some regions of the adjoint space may be inaccessible, it will be preferable to reverse the procedure of section (1), and instead carry discussion of the D-term into the fundamental space.

The rank of $SO(2n)$, or number of mutually commuting generators, is equal to $n = \frac{m}{2}$, the number of independent rotation planes, or one half the spatial dimension. Each diagonal matrix has eigenvalues of $(+1, -1)$, with the remaining entries all zero. The non-diagonal matrices can be combined into raising and lowering operators of unit strength, always acting in two diagonal sectors while leaving the rest unaffected. Every possible pairing of sectors and choice of raising or lowering in each sector is represented. Another counting exercise, with the $\binom{n}{2}$ ways to pick two diagonal generators, times a factor of 4 for the choices $\begin{pmatrix} + & + & - & - \\ + & - & + & - \end{pmatrix}$, plus n for the diagonal matrices themselves, yields the correct total of $2n(n-1)+n = \frac{m(m-1)}{2}$, and verifies the consistency of this construction. The commutation relations between the operators obey a sort of charge conservation, with the net raising or lowering weight for each sector preserved across the equality. For SO(6), these matrices are shown explicitly in both the original and diagonal bases, along with example commutators, in the appendix.

Diagonalization of the secular equation $\det(M - \lambda \mathbb{1}) = 0$ for a matrix M, yields a product $\prod_{i=1}^{m}(\lambda_i - \lambda)$, whose roots specify the matrix eigenvalues, and whose expansion

$$(a_0 = 1)\lambda^m + a_1\lambda^{m-1} + a_2\lambda^{m-2} + \cdots + a_m = 0 \tag{3.25}$$

produces m coefficients a_i, which must be invariant under group transformations[4], as

[4]Any 'similarity' transformation which is enacted as $S^{-1}MS$ will preserve these

is the determinant they arise from. The coefficients are sums of products of eigenvalues, e.g. $a_1 = \sum_{i=1}^{m} \lambda_i$, $a_2 = \sum_{i>j=1}^{m} \lambda_i \lambda_j$, $a_3 = \sum_{i>j>k=1}^{m} \lambda_i \lambda_j \lambda_k$, and $a_m = \prod_{i=1}^{m} \lambda_i$, which provides concrete verification for the claim of invariance. These factors, equivalent to a full knowledge of the eigenvalues, fully encode all the rotationally invariant properties of M, and fully specify all forms into which M may be rotated. However, the specific combinations shown prove to have their own desirable properties. For example a_1 and a_m may be recognized as the familiar trace and determinant. In general, all m coefficients may be constructed as a function only of the matrix M, by use of the recursive form

$$a_j = \frac{1}{j} \sum_{i=1}^{j} (-1)^{i+1} \mathrm{Tr}\,(M^i) a_{j-i}\,, \tag{3.26}$$

referenced to the starting value $a_0 \equiv 1$. For the generators of $SO(2n)$, the odd-valued coefficients vanish automatically, due to the tracelessness (antisymmetry) of odd powers. It is important to note that this leaves only n constraints (equal to the group rank) to be actively satisfied in any basis change of the generators. For $SO(6)$, we have specifically:

$$a_2 = \frac{-\mathrm{Tr}\,M^2}{2},\ a_4 = \frac{-\mathrm{Tr}\,M^4 - a_2 \mathrm{Tr}\,M^2}{4},\ a_6 = \frac{-\mathrm{Tr}\,M^6 - a_2 \mathrm{Tr}\,M^4 - a_4 \mathrm{Tr}\,M^2}{6}\,. \tag{3.27}$$

Similar to Eq. (3.25), is a theorem by Euler stating that $\prod_{i=1}^{m}(M - \lambda_i \mathbb{1}) = 0$, as an operation on any of the m $|\lambda_i\rangle$ must be null. This equation may, in principle, be used to reduce by one, with each application, the highest power of a series in the matrix M, until the limiting case where that power is $m-1$. The ability to envision this procedure justifies the statement that the Taylor expansion for a function of M is truncated to order $m-1$. This knowledge is critical to studying the finite rotation

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invariants. Orthogonal and unitary mappings are the most notable examples.

operator

$$e^{-i\Theta_{ab}M_{ab}}\,. \tag{3.28}$$

We will focus initially on the case of just a single rotation plane and angle θ in 6-space, with the adjoint unit vector $\hat{n}$ providing our combination of generators, $\theta(\vec{M}\cdot\hat{n})$. An explicit representation, consistent with Eq. (3.23), of the generic planar generator can be formed readily, where $\hat{a}$ and $\hat{b}$ are orthogonal unit 6-vectors and the notation $\otimes$ is defined by the construction of a tensor, "$|\ \rangle\langle\ |$", out of the column to its left and the row to its right.

$$\vec{M}\cdot\hat{n} \Leftrightarrow M_{\hat{a}\hat{b}} \equiv i(\hat{a}\otimes\hat{b} - \hat{b}\otimes\hat{a}) \tag{3.29}$$

It is clear from this construction that all single plane generators obey the rule

$$(\vec{M}\cdot\hat{n})^2 \Leftrightarrow M_{\hat{a}\hat{b}}\cdot M_{\hat{a}\hat{b}} = (\hat{a}\otimes\hat{a} + \hat{b}\otimes\hat{b})\,, \tag{3.30}$$

meaning that $(\vec{M}\cdot\hat{n})^3 = \vec{M}\cdot\hat{n}$, and $(\vec{M}\cdot\hat{n})^4 = (\vec{M}\cdot\hat{n})^2$, etc. and thus that the rotation operator reduces further in this case, to only 3 terms with 3 undetermined coefficients (α,β,γ).

$$e^{-i\theta(\vec{M}\cdot\hat{n})} = \alpha + \beta(\vec{M}\cdot\hat{n}) + \gamma(\vec{M}\cdot\hat{n})^2 \tag{3.31}$$

Not coincidentally, this is also the number of available discrete eigenvalues, and forcing consistency in Eq. (3.31) when $\vec{M}\cdot\hat{n}$ is replaced by $(+1,0,-1)$, allows us to fix the parameters shown:

$$e^{-i\theta(\vec{M}\cdot\hat{n})} = \mathbb{1} - i\sin\theta(\vec{M}\cdot\hat{n}) + (\cos\theta - 1)(\vec{M}\cdot\hat{n})^2\,. \tag{3.32}$$

With this reduced case in hand, we can now return attention to the general finite $SO(2n)$ transformation specified by the linear combination of rotation generators

$\Theta_{ab}M_{ab}$. To start, we will write the prototype block diagonalized form of a member of this class with n free angles θ_i.

$$i \times \begin{pmatrix} \begin{pmatrix} 0 & \theta_1 \\ -\theta_1 & 0 \end{pmatrix} & & \\ & \begin{pmatrix} 0 & \theta_2 \\ -\theta_2 & 0 \end{pmatrix} & \\ & & \ddots \end{pmatrix} \tag{3.33}$$

Just as the operator of Eq. (3.28) enacts changes of basis on real $2n$-vector states, *orthogonal pairs* of such operators invoke the corresponding similarity (rank-2 tensor) transformation on *elements of* $SO(2n)$ *itself*[5], i.e. specific *instances of* Eq. (3.28). In the Taylor sense, the same transformation is then applied to the selected generators, transforming them into the alternate basis[6].

$$(\Theta_{ab}M_{ab})' \equiv e^{-i\Phi_{cd}M_{cd}}(\Theta_{ab}M_{ab})e^{+i\Phi_{cd}M_{cd}} \tag{3.34}$$

Rotation can only map a linear sum of $SO(2n)$ generators into *another* such linear sum. Preservation of the m rotational invariants is the only restriction on what members of this class may be interrelated by operation of $SO(2n)$. As is necessarily expected, orthogonal transformations explicitly protect the property of (anti)symmetry, ensuring that the odd invariants remain identically zero.

$$(M')^T \equiv (OMO^{-1})^T = OM^TO^T = (-)M' \tag{3.35}$$

This leaves then only n non-trivial constraints, which should always be absorbable via some action of the group *itself*, into the n angles of Eq. (3.33). Specifically, for

[5]This ensures then that the same result is achieved whether the change of basis occurs before or after operation of the $SO(2n)$ element on a given vector.

[6]Or alternatively, performing an opposing rotation while the basis stays fixed, depending on taste.

the case of $m = 6$, we have:

$$\begin{aligned} a_2 &= -(\theta_1^2 + \theta_2^2 + \theta_3^2) \\ a_4 &= ((\theta_1\theta_2)^2 + (\theta_2\theta_3)^2 + (\theta_3\theta_1)^2) \\ a_6 &= -(\theta_1\theta_2\theta_3)^2 \end{aligned} \tag{3.36}$$

The solubility[7] of equations like the above is identical to the statement that any matrix $\Theta_{ab}M_{ab}$ may in principle be converted to the desired block-diagonal form under operation of $SO(2n)$. Note that in each term angles ever appear individually only as squares.

Since the commutativity of these n sectors from (3.33) will be maintained under any group action, it may be inferred that all combinations of generators are a sum over n orthogonal rotation planes. Thus, the finite rotation operator in (3.28) can be factored into separate exponential terms containing each distinct plane, without any complications of the Baker–Hausdorff variety. It is then seen that the general transformation is enacted by a *product* of n (3.32) copies.

Note that this *does not* imply that $SO(2n)$ is equivalent to the factored product $O(2)^n$. Rather, it simply says that for *each* rotation in $SO(2n)$ which you would like to perform there is in principle an identical representation for that *particular* operation in terms of a product of planar rotations.

2. Viewing D-Terms From the Fundamental Space

Having understood the $SO(2n)$ group structure to some degree, the next matter which we may wish to consider is the manner in which field VEVs of the fundamental

[7] Mathematica can readily invert the example system. However, the general solutions are quite clumsy in appearance.

space are transformed into D-terms of the adjoint space. Ultimately we will also be interested in how the concisely realized adjoint D-constraint, namely that the sum of all vector contributions be null, is reflected back onto fundamental states. It will be useful first to determine the eigenvectors corresponding to generators of rotation in a given plane, such as appear in (3.29). Within an arbitrary phase, the normalized solutions for the $(+1,-1)$ eigenvalues of $M_{\hat{a}\hat{b}}$ may be written respectively as:

$$|+\rangle_{\hat{a}\hat{b}} = (\hat{a} - i\hat{b})/\sqrt{2}\,,\ \ |-\rangle_{\hat{a}\hat{b}} = (\hat{a} + i\hat{b})/\sqrt{2}\,. \tag{3.37}$$

As expected for a hermitian matrix, these states are mutually orthogonal ($\langle +|-\rangle_{\hat{a}\hat{b}} = 0$), and may also be chosen with null projections onto the $2n-2$ additional states required to complete the basis. They are not generally orthogonal to the eigenvectors of overlapping generators, such as "$M_{a\hat{c}}$", which induces rotation in a plane passing through $\hat{a}$ and some vector $\hat{c}$, where $\hat{b}\cdot\hat{c} = 0$. It is interesting to note that the real generator basis as established in (3.23) produces intrinsically complex eigenvectors, while the diagonal matrix set described in the Appendix is instead itself complex with the possibility of real eigenvectors. These two formulations are bridged by a unitary transformation rather than any $SO(2n)$ element.

The 'expectation value' of an eigenstate contracted on its corresponding matrix generator is given by:

$$\langle +|M_{\hat{a}\hat{b}}|+\rangle \equiv \frac{i}{2}(\hat{a} + i\hat{b})\cdot(\hat{a}\otimes\hat{b} - \hat{b}\otimes\hat{a})\cdot(\hat{a} - i\hat{b}) = 1\,. \tag{3.38}$$

The same calculation performed with the $|-\rangle_{\hat{a}\hat{b}}$ state will yield -1. We can argue that all other contractions employing the $|\pm\rangle_{\hat{a}\hat{b}}$ are vanishing since the other $n-1$ 'diagonal' generators work in orthogonal spatial sections, and the $2n(n-1)$ 'raising and lowering' operators cannot bridge an eigenstate to itself. Thus, the complete

$2n(2n-1)$ element D-term which emerges out of this state can be determined:

$$|\frac{\hat{a}-i\hat{b}}{\sqrt{2}}\rangle \implies (\hat{a}\otimes\hat{b}-\hat{b}\otimes\hat{a})_{ij}\,. \tag{3.39}$$

Notice that while previously in (e.g. Eq. 3.29) a similar notation was used to express the matrix elements of a *single* generator $M_{\hat{a}\hat{b}}$, the same form now provides the 'expectation value' result for *every* generator, interpreted as either an adjoint vector ($i>j$), or as a fundamental antisymmetric matrix. For example, if we take $\hat{a}=(1,0,0,\cdots)$ and $\hat{b}=(0,1,0,\cdots)$, then (3.39) says that the related complex $\mathbf{2}n$ dimensional vector will produce a D-term contribution of $\langle M_{12}\rangle = 1(=-\langle M_{21}\rangle)$, with all other elements zero. Covariance suggests that the same expression must also hold for vectors $(\hat{a},\hat{b})$ which do not lie along a single direction of the selected basis.

We may next wish to inquire what form the D-term matrix would take for a fully arbitrary $\mathbf{2}n$-plet of VEVs, *i.e.* with the constraints $\hat{a}\cdot\hat{a}=\hat{b}\cdot\hat{b}=1$ and $\hat{a}\cdot\hat{b}=0$ relaxed. To avoid confusion with the previous results, we will now refer to the unit vectors $\hat{c}$ and $\hat{d}$, used to compose our general state as:

$$|v\rangle \equiv R\hat{c}+iI\hat{d}\,. \tag{3.40}$$

We can still insist without any loss of generality that the state be normalized to unity overall, and in fact this condition facilitates a simple rescaling to integral multiples of the squared FI scale $|\langle\alpha\rangle|^2$ as is typically required. With this condition in place, the real coefficients R and I are restricted to $R^2+I^2=1$.

In correspondence to the earlier (3.37), we can also define the states

$$|\pm\rangle_{\hat{c}\hat{d}} \equiv (\hat{c}\mp i\hat{d})/\sqrt{2}\,, \tag{3.41}$$

although a word of caution is in order to the effect that *no* eigenvalue relation to some matrix $M_{\hat{c}\hat{d}}$ is being posited. In fact, it is no longer even necessarily true that

$\langle +|-\rangle_{\hat{c}\hat{d}} = 0$ However, in terms of these states there is an alternate formulation of the general complex state:

$$|v\rangle = A|+\rangle_{\hat{c}\hat{d}} + B|-\rangle_{\hat{c}\hat{d}}\,, \tag{3.42}$$

$$R = \frac{A+B}{\sqrt{2}}\ ,\ I = -\frac{A-B}{\sqrt{2}}\,. \tag{3.43}$$

The normalization condition is here realized as $A^2 + B^2 = 1$. With these conventions in place the expectation value on any generator M can now be decomposed.

$$\langle v|M|v\rangle = R^2\langle c|M|c\rangle + I^2\langle d|M|d\rangle + iRI(\langle c|M|d\rangle - \langle d|M|c\rangle) \tag{3.44}$$

We will now focus on the term $\langle c|M|c\rangle$. Since this is a scalar contraction, the transpose operator will be an identity. Furthermore, since the vector $\hat{c}$ is real by definition, the state $\langle c|$ is merely $|c\rangle^T$, so that $\langle c|M|c\rangle = \langle c|M^T|c\rangle$. However, the antisymmetry of M says that this expression is equal to its own negative, and must therefore vanish. The same result holds for $\langle d|M|d\rangle$, and in fact for *any* D-term constructed from a purely real vector[8]. In a sign of consistency, the remaining two terms are proportional to "RI", such that all D-term contributions vanish unless $|v\rangle$ contains *both* real and imaginary contributions. An argument similar to that just given on the transpose suggests that these terms are opposites. Also, since they are clearly related by complex conjugation, this means that each term is purely imaginary, preserving the necessary reality condition on the expectation value. Combining the information above with Eqs. (3.41, 3.43) allows us to state that:

$$\langle v|M|v\rangle = 2iRI\langle c|M|d\rangle = (A^2 - B^2)\langle +_{\hat{c}\hat{d}}|M|+_{\hat{c}\hat{d}}\rangle\,. \tag{3.45}$$

[8]This conclusion is also true for 'trivially complex' vectors, as the inclusion of an *overall* complex phase can be absorbed in the contraction without altering the remaining discussion.

The normalization imposed on the A and B coefficients restricts the pre-factor $A^2 - B^2$ to range between (-1, 1). The lesson to be taken from these exercises is that an unbalanced mixing between the 'I' and 'R' coefficients of a given fundamental state will not effect the *orientation* of the adjoint space D-term which it yields, but it can reduce the *scale* of that contribution.

However, to complete analysis of the general $\mathbf{2}n$ vector, there remains the matter of $\hat{c} \cdot \hat{d} \neq 0$ to deal with. In order to proceed, let us introduce an alternate unit vector $\hat{d'}$ which is perpendicular to $\hat{c}$ and lies in the plane defined by $\hat{c}$ and the original $\hat{d}$. Breaking $\hat{d}$ down into its new basis components, we have:

$$\hat{d} = \hat{d'} \sin\theta + \hat{c} \cos\theta \,, \tag{3.46}$$

where θ is the angle separating $\hat{c}$ and $\hat{d}$.

Under this transformation, the state $|v\rangle$ from (3.40) becomes

$$\begin{aligned} |v\rangle &= R\hat{c} + iI(\hat{d'} \sin\theta + \hat{c}\cos\theta) && (3.47) \\ &= \frac{R + iIe^{i\theta}}{\sqrt{2}} |+\rangle_{\hat{c}\hat{d'}} + \frac{R + iIe^{-i\theta}}{\sqrt{2}} |-\rangle_{\hat{c}\hat{d'}} \,. && (3.48) \end{aligned}$$

The main points to notice here are that we can decompose $|v\rangle$ into the orthogonal eigenstates of the pure generator $M_{\hat{c}\hat{d'}}$, and that no single raising or lowering operator can hope to join these states[1]. Because of this, the resultant D-term will again be fully along the $M_{\hat{c}\hat{d'}}$ orientation, and we have only to find its magnitude. Since the $|\pm\rangle_{\hat{c}\hat{d'}}$ coefficients in (3.48) are complex, the expression does not form a realization of (3.42), nor should the results following that equation be expected to apply[2].

[1] The $|\pm\rangle_{\hat{c}\hat{d'}}$ are separated by a third 'rung' corresponding to some state with a null response to the ($\hat{c}$,$\hat{d'}$) plane rotation operator.

[2] Although (3.42) did in fact represent the *general* $\mathbf{2}n$-plet, this was for $\hat{c}$ and $\hat{d}$ unconstrained. Since a definite condition between $\hat{c}$ and $\hat{d'}$ has now been imposed,

Rather than proceeding directly with (3.48), let us instead evaluate $\langle c|M_{\hat{c}\hat{d}'}|d\rangle = i\langle +_{\hat{c}\hat{d}}|M_{\hat{c}\hat{d}'}|+_{\hat{c}\hat{d}}\rangle$, as appears in (3.45). Rewriting (3.46) as

$$\hat{d}' = \frac{\hat{d} - \hat{c}\cos\theta}{\sin\theta}\,, \tag{3.49}$$

it is readily confirmed that:

$$M_{\hat{c}\hat{d}'} \equiv i(\hat{c}\otimes\hat{d}' - \hat{d}'\otimes\hat{c}) = i\frac{(\hat{c}\otimes\hat{d} - \hat{d}\otimes\hat{c})}{\sin\theta}\,. \tag{3.50}$$

The desired contraction then becomes

$$\langle c|M_{\hat{c}\hat{d}'}|d\rangle = i\frac{\hat{c}\cdot(\hat{c}\otimes\hat{d} - \hat{d}\otimes\hat{c})\cdot\hat{d}}{\sin\theta} = i\frac{1-\cos^2(\theta)}{\sin\theta} = i\sin\theta\,. \tag{3.51}$$

Combining this result with (3.45) allows us to summarize the magnitude of the general D-term as

$$|v\rangle \equiv R\hat{c} + iI\hat{d} \implies |D_v| = -2RI\sin\theta = (A^2 - B^2)\sin\theta\,. \tag{3.52}$$

where θ is the angle between $\hat{c}$ and $\hat{d}$, and the relation appearing in (3.42) is also in use. Extending the intuition that only non-trivially complex states correspond to D-terms, the $\sin(\theta)$ term will kill $|D_v|$ if $\hat{c}$ and $\hat{d}$ are proportional, such that $|v\rangle$ is real times an overall phase. Also, we can note that this scale is exactly sufficient when integrated with the known adjoint direction of (3.50) to exactly mimic the simple result of (3.39) [3] for the total D-term. Specifically, under the redefinitions $\vec{e} \equiv \sqrt{2}R\hat{c}$ and $\vec{f} \equiv -\sqrt{2}R\hat{d}$, the general result becomes

$$|v\rangle \equiv |\frac{\vec{e} - i\vec{f}}{\sqrt{2}}\rangle \implies (\vec{e}\otimes\vec{f} - \vec{f}\otimes\vec{e})_{ij}\,. \tag{3.53}$$

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complex coefficients are required for the general θ.

[3]Recall, as per the discussion following this equation, that the *orientation* corresponding to a state which strikes only a single generator may be written in matrix form as the generator *itself*, dropping the factor of i.

Basic manipulations are enough to reveal an additional equivalent construct:

$$(D_v)_{ij} \equiv \langle v|M_{ij}|v\rangle = (\frac{|v\rangle\langle v| - |v^*\rangle\langle v^*|}{i})_{ij}\,. \tag{3.54}$$

Finally, we come to the interesting realization that, having exhausted all available generality in our state $|v\rangle$, and receiving still for this effort only D-terms which correspond to single-plane rotation generators as in (3.53), it is *not possible* to represent an arbitrary adjoint space direction by the VEVs of a solitary $\mathbf{2}n$-plet. This conclusion is born out by analogy to the discussion around (3.33), where it was shown that the general $SO(2n)$ transformation matrix, *i.e.* adjoint space vector, required contributions from n orthogonal planar type generators times scaling angles.

3. An Example of D-Flatness in $SO(6)$

To be more concrete, let us specifically consider D-flat directions formed from the VEVs of fundamental vector $\mathbf{6}$ representations for the gauge group $SO(6)$, choosing to express the generators in their 'original' basis as shown in the Appendix. We will denote the (generally complex) VEVs of the k^{th} 6-plet according to,

$$< \mathbf{6}_k >= \begin{pmatrix} \alpha_{k,1} + i\,\beta_{k,1} \\ \alpha_{k,2} + i\,\beta_{k,2} \\ \alpha_{k,3} + i\,\beta_{k,3} \\ \alpha_{k,4} + i\,\beta_{k,4} \\ \alpha_{k,5} + i\,\beta_{k,5} \\ \alpha_{k,6} + i\,\beta_{k,6} \end{pmatrix}, \tag{3.55}$$

where $\alpha_{k,j}$ & $\beta_{k,j}$ are real. D-flatness constraints when applied to a single 6-plet (with field subscript $k = 1$) demand the vanishing of:

$$D_{ij} \equiv (\alpha_{1,a} + i\,\beta_{1,a}) \cdot \{\delta_i^a \delta_j^b - \delta_i^b \delta_j^a\} \cdot (\alpha_{1,b} + i\,\beta_{1,b})\,, \tag{3.56}$$

as can be enforced by direct computation from (3.23). Contracting indices for a more concisely worded expression,

$$\{\alpha_{1,i}\beta_{1,j} - \alpha_{1,j}\beta_{1,i} = 0\}\,, \tag{3.57}$$

for $i, j = 1$ to 6 and $i < j$. This is of course equivalent to a vanishing of the construct from (3.53).

The only solutions to (3.57) are fundamentally trivial. Since real and imaginary VEV component coefficients, α and β, always appear together as products, a solution exists for any pure real or pure imaginary vector. However, even for those solutions which employ cancellation of D-term contributions between $\alpha_{1,i}\,\beta_{1,j}$ products, D-flatness is maintained if and only if the ratio $\alpha_{1,j}/\beta_{1,j}$ is the same for all non-vanishing members j. Furthermore, neither $\alpha_{1,j} \neq 0$ & $\beta_{1,j} = 0$ nor $\alpha_{1,j} \neq 0$ & $\beta_{1,j} = 0$ is allowed for any j [86]. This statement of constant phase is in agreement with the statements on 'reality conditions' from the previous section, specifically as appears following (3.44, 3.52).

It is somewhat more interesting to generalize the above solutions for a single $< \mathbf{6} >$ to a set of n distinct $\mathbf{6}$'s, each having respective real and imaginary VEV components $\alpha_{k,j} + i\,\beta_{k,j}$ for $k = 1$ to n. The constraints become a sum over expressions of the sort in (3.57),

$$\{\sum_{k=1}^{n} \alpha_{k,i}\beta_{k,j} - \alpha_{k,j}\beta_{k,i} = 0\}\,, \tag{3.58}$$

for $i, j = 1$ to 6 and $i < j$. Rather than forcing reality on just one state, we instead now have a dispersement of the burden of D-flatness among every constituent field, none of which need individually follow the former condition. Each new 6-plet VEV generates further non-trivial solution classes, allowing new possibilities for cancellations between different fields, and additional freedoms for each individual state. The simplest non-trivial example involves two fields. If each field is further restricted to non-zero VEVs for only its top two complex components, that is, for $i, j = 1, 2$, then the $D-$flat constraints (3.58) reduce to simply

$$\alpha_{1,1}\beta_{1,2} - \alpha_{1,2}\beta_{1,1} + \alpha_{2,1}\beta_{2,2} - \alpha_{2,2}\beta_{2,1} = 0\,. \tag{3.59}$$

Hence we are free to choose seven of the eight VEV components.

4. Reduction and Interpretation of the Multiple-VEV Constraint

The condition written in (3.58) is clearly a full specification of the desired D-term result, but it cannot fulfill the *spirit* of our search, in that the expression is neither geometrical nor intuitively comprehensible. In fact, the number of conditions enforced grows like the group adjoint dimension, and quickly becomes so large that even inelegant and forceful approaches may find the task of its solution insuperable. We would much prefer a statement of principle from which one could deduce at a glance the *existence* of solutions, followed in short order by *specific* field values obedient to the criterion. Furthermore, the prospect of a condition whose complexity grows *linearly* with the group rank n, rather than *quadratically* is enticingly motivated by the earlier observation that any adjoint vector can be decomposed into n commuting anti-symmetric fundamental matrices of the special rotation generator form, (3.29). This section will then pursue results useful for understanding D-flatness in the presence of multiple $SO(2n)$ VEVs.

We can proceed by denoting each contributing state as $|\gamma\rangle$, so that the net D-term will be:

$$D_{ij} \equiv \sum_{\gamma} \langle\gamma|M_{ij}|\gamma\rangle\,. \tag{3.60}$$

As discussed around (3.33), this matrix can be viewed as the sum of n orthogonal single-plane rotation matrices times scale factors. We know from (3.39) that any such matrix can be composed by the contraction of its 'positive' eigenvector around M_{ij}. Furthermore, the independence of each of the n matrices tells us that their eigenvectors will also be eigenvectors of the total D-term matrix in (3.60), which will be written as $|\lambda\rangle$. Therefore, we can state the following relationships between the D-matrix and its eigenvectors:

$$\sum_{\gamma} \langle\gamma|M_{ij}|\gamma\rangle = \sum_{k=1}^{n} \lambda_k^+ \langle\lambda_k^+|M_{ij}|\lambda_k^+\rangle = \sum_{k=1}^{2n} \lambda_k (|\lambda_k\rangle\langle\lambda_k|)_{ij} \,. \tag{3.61}$$

The front factor of λ_k^+ is needed in the central expression to represent the scale factor, which shifts the eigenvalue away from unity, of each 'planar' member in the composite D-term[4]. Since M_{ij} is Hermitian and complex, we can see that its eigenvalues always come in $\pm\lambda$ pairs, with eigenvectors related by complex conjugation.

$$M|\lambda\rangle = \lambda|\lambda\rangle \implies M|\lambda^*\rangle = (M^*|\lambda\rangle)^* = -(M|\lambda\rangle)^* = -\lambda|\lambda^*\rangle \,. \tag{3.62}$$

Also noting that

$$\langle\gamma|M_{ij}|\gamma\rangle = -\langle\gamma^*|M_{ij}|\gamma^*\rangle \,. \tag{3.63}$$

for any $|\gamma\rangle$, it is clear that the plus sign on λ_k^+ should really just be taken to indicate that half of the eigenvalues (one from each pair) are in play here, with a symmetry protecting the choice of either the 'positive' or 'negative' member. This is in accord with the properties previously observed directly in section (2). The third expression in (3.61) can be readily justified from the second, via comparison to Eqs. (3.53, 3.54),

[4]In other words, the contraction of $|\lambda\rangle$ around M_{ij} reconstructs the given constituent of D_{ij} with an overall unit scale, but since $|\lambda\rangle$ *is* the eigenvector of the matrix to be rebuilt, its eigenvalue λ and the desired scale are in fact one and the same.

when written as:

$$\sum_{k=1}^{n} \lambda_k^+ (|\lambda_k^+\rangle\langle\lambda_k^+| - |\lambda_k^*\rangle\langle\lambda_k^*|)_{ij} = \sum_{k=1}^{n} \lambda_k^+ (\hat{a_k} \otimes \hat{b_k} - \hat{b_k} \otimes \hat{a_k})_{ij} \,, \tag{3.64}$$

where $|\lambda_k^+\rangle \equiv (\hat{a_k} - i\hat{b_k})/\sqrt{2}$. This alternate method of writing the same sum of matrices should be recognized as simply a diagonalization by similarity transformation[5].

We can now read directly from (3.61) that the n vectors represented as $\sqrt{\lambda}|\lambda\rangle$ function as a completely equivalent set of input to the original fully general $|\gamma\rangle$'s. This reinforces the notion that it may be possible to impose some reduced number of conditions equivalent to the rank which will serve to eliminate all $n(2n-1)$ elements of the D-matrix. The only flaw in this approach is our complete inability to know $|\lambda\rangle$ *before* we have *already* specified the overall VEV set!

From another perspective, we might imagine that a concrete expression for the eigenvalues $|\lambda\rangle$ of the matrix from (3.60) would enable a clear view of what conditions ensure that these numbers will vanish. By inserting dual complete sets, each written as $\mathbb{1} \equiv \sum |\lambda\rangle\langle\lambda|$, into the first expression of (3.61), we can say:

$$\sum_{k,l=1}^{2n} \langle\lambda_k|M_{ij}|\lambda_l{}'\rangle \left(\sum_{\gamma} \langle\gamma|\lambda_k\rangle\langle\lambda_l{}'|\gamma\rangle \right) = \sum_{k=1}^{n} \lambda_k^+ \langle\lambda_k^+|M_{ij}|\lambda_k^+\rangle \,. \tag{3.65}$$

The strength of these equations in the (i,j) is enough to let us equate coefficients of each contraction across the two sides. This is seen most cleanly in a basis where M_{ij} is diagonal, such that 'cross-terms' on the left-hand side of (3.65) vanish[6], and

[5]By starting with the conception of a diagonal form, one could in fact read this argument in reverse as an alternate proof of the decomposition into n planes.

[6]We are referring here to the n diagonal matrices; the remaining raising and lowering operators can join unmatched kets, but the sum of all such terms must vanish since these elements make no contribution on the left.

we are left with

$$\sum_{\gamma}\left(|\langle\gamma|\lambda^+\rangle|^2 - |\langle\gamma|\lambda^*\rangle|^2\right) = \lambda^+\,, \tag{3.66}$$

also using (3.63). Depending on your preference, this expression may be regrouped into various forms with distinct interpretations.

$$\lambda = \sum_{\gamma}\langle\gamma|\left(|\lambda\rangle\langle\lambda| - |\lambda^*\rangle\langle\lambda^*|\right)|\gamma\rangle = \langle\lambda|\sum_{\gamma}\left(|\gamma\rangle\langle\gamma| - |\gamma^*\rangle\langle\gamma^*|\right)|\lambda\rangle\,, \tag{3.67}$$

The first of these constructions appears as an element of (3.60), and again validates the notion that the entire matrix could be set to zero via consideration of only a reduced subset of n 'diagonal' generators *if only* one could know ahead of time which diagonal set to choose! We can also see that each eigenvalue of the overall D-matrix vanishes in turn when its corresponding eigenvector is 'real', up to an overall phase. This is natural in light of the discussion above where it was noted that the $\sqrt{\lambda}|\lambda\rangle$ serve as equivalent input to the VEVed $|\gamma\rangle$'s[7], and that their eigenvalues flip sign under complex conjugation. Another interpretation of this expression holds that λ is the imbalance between projections onto the 'positive' and 'negative' eigenvectors. The second formulation of (3.67) is arrived at by applying the (free) operation of complex conjugation to the (real) second element of (3.66) before separating out the terms. From this, we can read another condition, applied now to the more tangible input states, which will also kill the D-term:

$$\sum_{\gamma}|\gamma\rangle\langle\gamma| \implies \text{REAL}\,. \tag{3.68}$$

By way of first analysis, we can note that this expression is immune to overall phase factors associated with $|\gamma\rangle$, and that it properly reduces to the 'generalized reality' condition imposed on just a single state. In fact though, the statement of (3.68) con-

[7]As pointed out around (3.52), real SO(2n) states create no D-term contributions.

tains only the same information, and the same shortcomings of the previous attempts. Using (3.54), the entire D-term can be written

$$D_{ij} = \sum \gamma(|\gamma\rangle\langle\gamma| - |\gamma^*\rangle\langle\gamma^*|)_{ij}\,, \tag{3.69}$$

and (3.68) is simply the statement that the matrix should vanish[8]. Alternatively, using the language of (3.53), we can say that

$$\sum_{\gamma}(\vec{e}_\gamma \otimes \vec{f}_\gamma - \vec{f}_\gamma \otimes \vec{e}_\gamma)_{ij} = 0\,, \tag{3.70}$$

where $|\gamma\rangle \equiv (\vec{e}_\gamma - i\vec{f}_\gamma)/\sqrt{2}$. However, this only serves to mimic (3.58) and its associated large number of conditions for projections in each of the generator planes.

A fully geometric interpretation of non-Abelian $SO(2n)$ D-flat directions should offer proper, concise criterion with physically intuitive interpretation. Along this line, we have made several arguments for the feasibility of solutions that grow only like the group rank for increasing number of fundamental **2n** field VEVs. Realization of these arguments, nevertheless, remains an open issue, for after a detailed study, a more functional re-expression of (3.58) has not been found.

5. Ensuring Simultaneous $SO(2n)$ F-Flatness

For the previous study of $SU(2)$, computations were performed in the adjoint space such that the D-condition was readily realized, while effort was required to transfer F-terms into the language of orientations in this space (cf. Eq. 3.18). In contrast, we have worked up to this point in the fundamental space. The exertion of expressing D-terms via only their corresponding state VEVs is rewarded by a natural and straightforward

[8]The embedding of (3.69) within the second expression of λ in (3.67) is a consistency check, since the contraction of $|\lambda\rangle$ about the matrix which they represent must identically yield the eigenvalue.

implementation of F-flatness. However, even this ostensibly simple contraction is complicated in practice by the presence of distinct vector orientations for the real and imaginary VEVs. Having encountered severe obstacles in the transition to a purely fundamental description, this section will instead entertain something of a hybrid approach. Pragmatism will here overshadow the desire for generality, and an attempt will be made to extract from the previous technology some minimal working procedures to ensure simultaneous D- and F-flatness. New constructions will be added as needed.

There are three key quantities of interest in our search. These are the commutators between rotation generators, the scalar product between D-terms in the adjoint space, and the fundamental space VEV contraction. First, we will define here two fully general complex vectors to be used throughout the discussion. No restriction on the orthonormality of the constituent elements is assumed.

$$|\alpha\rangle \equiv (\vec{a} - i\vec{b})/\sqrt{2}\,, \quad |\gamma\rangle \equiv (\vec{c} - i\vec{d})/\sqrt{2} \tag{3.71}$$

Turning first to the commutator,

$$[M_{\vec{a}\vec{b}}, M_{\vec{c}\vec{d}}] = (i)^2 \left\{ (\vec{a} \otimes \vec{b} - \vec{b} \otimes \vec{a}) \cdot (\vec{c} \otimes \vec{d} - \vec{d} \otimes \vec{c}) - (\vec{c} \otimes \vec{d} - \vec{d} \otimes \vec{c}) \cdot (\vec{a} \otimes \vec{b} - \vec{b} \otimes \vec{a}) \right\}\,, \tag{3.72}$$

using the form of Eq. (3.29). This simplifies cleanly to

$$[M_{\vec{a}\vec{b}}, M_{\vec{a}\vec{b}}] = i \left\{ (\vec{a} \cdot \vec{d}) M_{\vec{b}\vec{c}} + (\vec{b} \cdot \vec{c}) M_{\vec{a}\vec{d}} - (\vec{a} \cdot \vec{c}) M_{\vec{b}\vec{d}} - (\vec{b} \cdot \vec{d}) M_{\vec{a}\vec{c}} \right\}\,, \tag{3.73}$$

which is equivalent to Eq. (3.24) in the orthonormal limit. This commutator vanishes if and only if the two rotations are fully disentangled, i.e. the rotation planes have a null intersection such that $(\vec{a},\vec{b})$ are mutually orthogonal to $(\vec{c},\vec{d})$. As before, the case that $(\vec{a},\vec{b})$[1] are themselves (anti)parallel is trivial, with $M_{\vec{a}\vec{b}} = 0$.

Next, we will examine the adjoint space contraction between the D-terms in correspondence to each of $|\alpha\rangle$ and $|\gamma\rangle$, denoted henceforth as $(\alpha{\cdot}\gamma)_A$. As established in Eq. (3.53), the needed adjoint space 'expectation values' are realized as the members of an antisymmetric matrix functionally identical, modulo the imaginary factor and a possible scale, to the rotation generator which would interpolate between vectors in the related fundamental space plane. To take the scalar product in this adjoint space, we must sum over the product of corresponding pairs between the two D-terms, including each of the $m(m-1)/2$ unique basis members one time. In terms of the provided form, this is realized as the trace of a matrix multiplication.

$$\begin{aligned}
(\alpha \cdot \gamma)_A &= -\frac{\alpha_{ij}\gamma_{ji}}{2} \\
&= -1/2 Tr \left\{ (\vec{a} \otimes \vec{b} - \vec{b} \otimes \vec{a}) \cdot (\vec{c} \otimes \vec{d} - \vec{d} \otimes \vec{c}) \right\} \qquad (3.74) \\
&= -1/2 Tr \left\{ (\vec{b} \cdot \vec{c}) \vec{a} \otimes \vec{d} + (\vec{a} \cdot \vec{d}) \vec{b} \otimes \vec{c} - (\vec{b} \cdot \vec{d}) \vec{a} \otimes \vec{c} - (\vec{a} \cdot \vec{c}) \vec{b} \otimes \vec{d} \right\}
\end{aligned}$$

A short diversion is in order here to investigate what is meant by the trace in this language. Recall that the notation $\vec{a} \otimes \vec{b}$ simply signifies the matrix constructed by the 'outer' product of the vectors $\vec{a}$ and $\vec{b}$. Thus:

$$\begin{aligned}
Tr(\vec{a} \otimes \vec{b}) &\equiv \begin{pmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 & \\ a_2 b_1 & a_2 b_2 & a_2 b_3 & \\ a_3 b_1 & a_3 b_2 & a_3 b_3 & \\ & & & \ddots \end{pmatrix} \\
&= a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots \equiv \vec{a} \cdot \vec{b} \qquad (3.75)
\end{aligned}$$

[1] Or, equivalently, $(\vec{c},\vec{d})$.

Armed with this knowledge, the expression from Eqs. (3.74) reduces nicely.

$$(\alpha \cdot \gamma)_A = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{b} \cdot \vec{c})(\vec{a} \cdot \vec{d}) \tag{3.76}$$

The next question of relevance is whether any geometrically intuitive representation is possible. We will make here a separation of the vectors $\vec{c}$ and $\vec{d}$ into two sections each, representing portions within and orthogonal to the $(\vec{a}, \vec{b})$ plane.

$$\vec{c} \equiv \vec{c}_{\parallel} + \vec{c}_{\perp} \ , \ \vec{d} \equiv \vec{d}_{\parallel} + \vec{d}_{\perp} \tag{3.77}$$

The orthogonal portions $(\vec{c}_{\perp}, \vec{d}_{\perp})$ trivially factor out of Eq. (3.76), reducing the analysis to a single plane. Furthermore, it is noted that the presence of all four vectors in both terms of this difference allow the extraction of an overall scale factor, leaving us to contend only with angles of orientation.

$$\begin{aligned} (\alpha \cdot \gamma)_A \quad &\Rightarrow \quad |a\, b\, c_{\parallel} d_{\parallel}| \left\{ \cos(\delta) \cos(\delta + \gamma - \alpha) - \cos(\alpha - \delta) \cos(\delta + \gamma) \right\} \\ &= \quad |a\, b\, c_{\parallel} d_{\parallel}| \sin(\alpha) \sin(\gamma) \end{aligned} \tag{3.78}$$

The angle[2] α separates $(\vec{a}, \vec{b})$, while γ splits $(\vec{c}_{\parallel}, \vec{d}_{\parallel})$, and δ is the angle between the vectors $(\vec{a}, \vec{c}_{\parallel})$. Basic trigonometric relations lead to the quite concise final result of Eq. (3.78), which has also a pleasing interpretation. Each of the pairs $(\vec{a}, \vec{b})$ and $(\vec{c}, \vec{d})$ correspond to an area within their plane constructed by completing the parallelogram which contains the vector pair as edges. Eq. (3.78) is the product of these areas, including only the portion which they project onto each other. As is required, for the case of $|\alpha\rangle = |gamma\rangle$, this result reduces to the square of Eq. (3.52)[3].

[2] When used in a trigonometric context a symbol such as α will designate the angle of separation between the real and imaginary constituents of the corresponding VEV state $|\alpha\rangle$.

[3] Note for comparison that $R \equiv \frac{a}{\sqrt{2}}$ and $I \equiv -\frac{b}{\sqrt{2}}$

We note that there are then three distinct mechanisms accessible for tuning the value of an adjoint scalar contraction. Firstly, the relative plane orientations can be tilted to effect a lesser or greater area of projection. It is clear that if only *one* of $(\vec{c}_{\|}, \vec{d}_{\|})$ vanishes, then the product is null. The condition $(\alpha \cdot \gamma)_A = 0$ is thus *weaker* than the statement $[M_{\vec{a}\vec{b}}, M_{\vec{c}\vec{d}}] = 0$, which requires complete independence of the two planes. Secondly, the internal angle between the real and imaginary portions of a single state can be adjusted to shorten or lengthen its overall D-term scale. Thirdly, one may of course consider rescaling the magnitude of the coefficients $|a\, b|$, although the combination $(a^2 + b^2)/2 = ||\alpha\rangle|^2$ is generally constrained in units of the squared FI-scale. In keeping with the discussion leading up to Eq. (3.53), these last two scenarios have the benefit of leaving *intact* the adjoint space *orientation*. The maximum adjoint space extension occurs when $|a| = |b|$ and $\vec{a} \cdot \vec{b} = 0$, in which case $|D| = a^2 = ||\alpha\rangle|^2$, which is proportional to the corresponding integral multiple of the fundamental scale.

The final quantity of interest for additional study here is the $SO(2n)$ invariant contraction in the fundamental space. This is simply the standard orthogonal inner product.

$$\begin{aligned} (\alpha \cdot \gamma)_F &\equiv \alpha^T \gamma \\ &\equiv \frac{1}{2}(\vec{a} \cdot \vec{c} - \vec{b} \cdot \vec{d}) - \frac{i}{2}(\vec{b} \cdot \vec{c} + \vec{a} \cdot \vec{d}) \end{aligned} \tag{3.79}$$

F. Concluding Remarks

We have observed the emergence of new techniques for the removal of dangerous terms from $\langle W \rangle$ and from $\langle F \rangle$. For example, four of the 'Table 1B' flat directions from [17] are lifted to all order by the vanishing of terms with more than two non-Abelian fields. One track suggested by the partial success of these flat directions is investigation of non-stringently flat directions for the FNY model that are flat to a finite order due to cancellation between various components in an F-term [86]. In this case, a benefit of the more difficult case of non-trivial D-flatness when compared to the simpler restriction of purely real state VEVs, which nevertheless ensure perfect compliance with D-flatness, may be that using generalized vectors can force flatness to be preserved order by order, so that when it eventually fails at some high order, a natural scale emerges at which supersymmetry is broken. It remains to be seen whether more attractive phenomenology with improved mass terms etc. may then result.

CHAPTER IV

INTERSECTING D-BRANE MODEL BUILDING

A. Structure of the Type IIA $T^6/\boldsymbol{Z}_4$ Orientifold

To extract phenomenologically viable models from a string context it is necessary to eliminate from observation the six 'extra' spatial dimensions which inhabit that theory. The two main approaches to solving this problem are the 'free fermionic' models, wherein new world sheet degrees of freedom are used instead of space-time coordinates to saturate the conformal anomaly, and the wrapping up of the undesirable dimensions on a suitable compact manifold. However, this ostensible plague on the string theory turns out to be a saving grace, in that all such procedures can translate the intrinsic symmetries of the extraneous space-time into interactions on gauged multiplets which naturally complement the ever-present Einstein gravity. The simplest such manifold which could be considered is the flat torus T^6. However, this turns out to be too naïve, preserving an inordinately large (maximal) amount of supersymmetry. The quest for $\mathcal{N} = 2$ $SUSY$ instead recommends extension to the curved 'Calabi-Yau' manifolds. Although there are examples of closed form model building in this context, the complexity of the treatment is daunting. Thankfully, there exist techniques which can yet salvage the humble torus by introducing a nontrivial topology via physical identification of points related modulo some discrete symmetry [87, 88, 89, 90, 91].

We will consider always that the torus T^6 is to be factored into three paired sets as $T^2 \times T^2 \times T^2$. Each two-torus is then represented on the complex plane as in the first element of Figure 7. Without loss of generality, the first toroidal edge may be scaled to length 1 and laid along the $\hat{x}$ axis. The second edge is conventionally taken

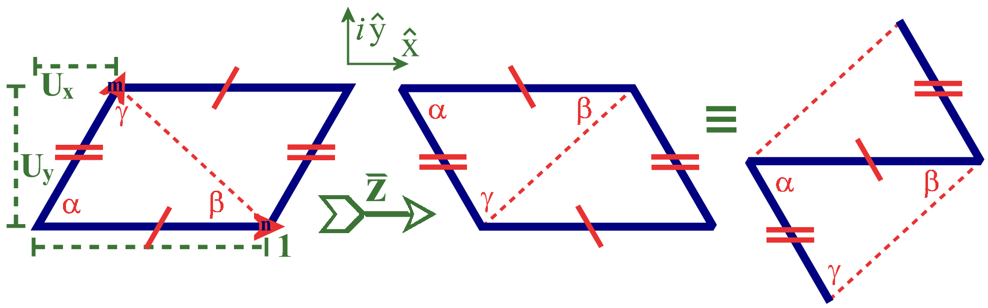

Fig. 7. *Anti-holomorphic Inversion of the Fundamental Torus* T^2

to lie in the upper half-plane with a length and tilting described by the 'complex structure' $U \equiv U_x + iU_y$. The prototype set of identifications imposed on the complex coordinates ($z = x + iy$) of a given T^2 are the 'orbifoldings' generated under action of the group $\mathbb{Z}_N$.

$$\mathbb{Z}_N: \qquad z \longrightarrow e^{\frac{i2\pi}{N}} z \tag{4.1}$$

The is represented diagrammatically in Figure 8 for the example $N = 6$. In general,

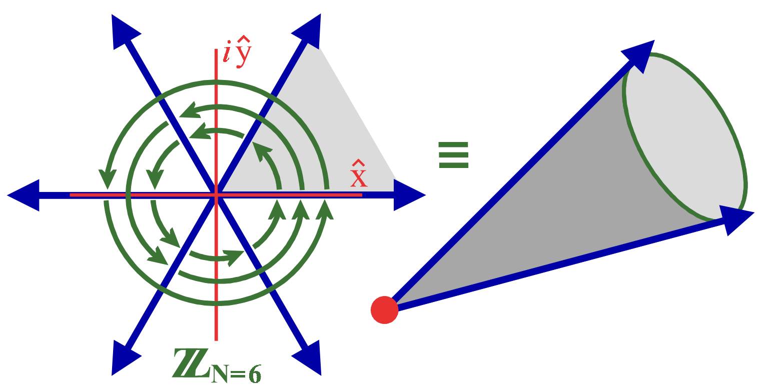

Fig. 8. *Action of the Symmetry* $\mathbb{Z}_N$ *in the Complex Plane*

$1/N^{th}$ of the plane becomes wrapped onto itself as a cone, creating a defect at the former origin. Strings can become entangled around such fixed points, thus developing trapped *orbits* about those positions. That the adjacent cell onto which one member identifies is in turn connected to its own neighbor is like unto the sequential application of the summed operators

$$\sum_{i=1}^{N} \Theta_N^i \,, \tag{4.2}$$

where Θ_N represents the generator of the $\mathbb{Z}_N$ transformation.

However, we are not dealing here with the extended complex plane, but instead a toroidally re-associated sector of that plane. If $\mathbb{Z}_N$ is to be a symmetry of the manifold, then it must act 'crystallographically' on the coordinates of the fundamental lattice cell. This means simply that the shape, scale and orientation of the torus are to be unaltered under the rotation. The scenario $\mathbb{Z}_4$ represents a 'counter-clockwise' rotation by 90 degrees. This is applied to the first two tori T^2, while the third goes through a conjugate treatment, 180° clockwise. The generator of this transformation will be labeled just Θ.

$$\Theta: \qquad z_1 \longrightarrow e^{\frac{i\pi}{2}} z_1 \ , \ z_2 \longrightarrow e^{\frac{i\pi}{2}} z_2 \ , \ z_3 \longrightarrow e^{-i\pi} z_3 \tag{4.3}$$

That this be true under 90° rotations restricts the shape of the first two tori strictly to be squares, that is $(U_x = 0; U_y = 1)$. The third T^2 on the other hand escapes unscathed, as all parallelograms map to themselves under rotation by π. The orbifolding of the square tori is graphically demonstrated in Figure 9. Straight (red) arrows represent translations between points identified under T^2, while curved (green) arrows are the rotations of $\mathbb{Z}_4$. The unit cell is truncated as expected to just one-quarter of its original area, shown as the shaded region. There are two fixed points (red dots) which map to themselves under these combined operations, located at a corner $(0,0)$ and center $(\frac{1}{2},\frac{1}{2})$ of the square. The remaining two (green) dots are instead interchanged by Θ. Diagonally opposed edges are similarly traded on the reduced

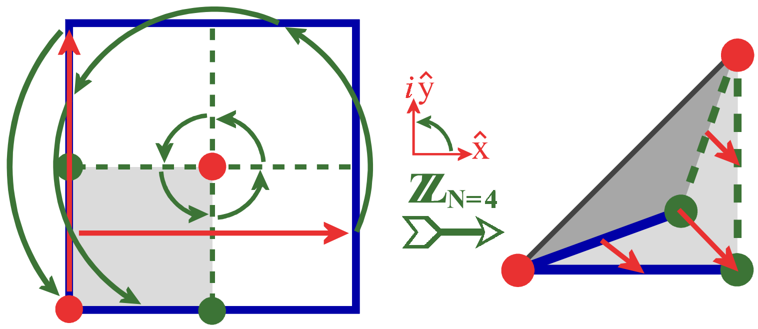

Fig. 9. *Effect of the Identification $\mathbb{Z}_4$ on the Square Torus T^2*

square, such that it folds into closure on itself as in the second element of Figure 9. Tori one and two are thus deformed from our traditional 'dough-nut' conception into the topology of a three cornered 'pillow'. The full set of singularities generated on T^6 are sufficient to break the supersymmetry which our manifold *supports* in the bulk to $\mathcal{N} = 2$[1]. This improved phenomenology is reflective of the fact that the $T^6/\mathbb{Z}_4$ orbifold *is actually* a Calabi-Yau threefold singular limit.

The next step in symmetry reduction is similar. Called the 'anti-holomorphic involution' $\bar{\sigma}$, it acts to identify points as:

$$\bar{\sigma}: \qquad z_i \rightarrow e^{i\phi_i}\bar{z}_i \tag{4.4}$$

The name of the game here is to lock down the available parameters such that this operation also preserves the torus. Let us look first at the square elements from the first and second T^2. The effect of the complex conjugation $\bar{z}$ is a simple reflection across the horizontal axis. The square retains its shape under this process so no additional rotation need be employed. Conceptually we are always free to shift points

[1]Strictly speaking, the induced topological defects break continuous differentiability and the orbifold is no longer a manifold at all.

along integral factors of the torus moduli, so we may imagine that the square is now translated back to its original home position. This has been called the **A** involution [20] of the first two tori. Alternatively though, taking $\phi_i = \frac{n\pi}{2}$ is equally satisfactory. The salient feature of this subtle change is that the invariant point set now lies diagonally across the torus, which will have tremendous ramifications on both the counting and supersymmetry of the allowed particle spectra. In conjunction with the repeated application of Θ, it is sufficient to consider only the "n=1" case, which will be called the **B** or 'tilted' involution.

$$\begin{aligned} \mathbf{A}_{1,2}: \qquad z &\rightarrow \bar{z} \\ \mathbf{B}_{1,2}: \qquad z &\rightarrow e^{\frac{i\pi}{2}}\bar{z} \end{aligned} \tag{4.5}$$

Treatment of the third torus is somewhat more delicate, and will follow Figure 7 closely. The $\bar{z}$ transformation is enacted on the 1^{st} diagram to arrive at the 2^{nd}. The 3^{rd} element of the figure shows an equivalent deformation by partial translation along the two moduli. Fixing any two of $(U_x, U_y, \alpha, \beta, \gamma)$ will fully determine the torus. First, it is clear that taking $\alpha = 90°$ (with U_y undetermined) is a special case. This will be the **A** involution on the third torus, wherein simple reflection without rotation is a proper symmetry. We are also allowed to transform by $\phi_3 = 180°$, but it is without supplementary effect. A rotation by $90°$ is generally excluded due to the elongated rectangular structure for $U_y \neq 1$; A second scenario of interest is $\alpha = \beta$, which has been labeled involution **B**. As shown in the 3^{rd} element of Figure 7, this can be reconfigured as a vertically oriented diamond, i.e. with the tips located at position $x = \frac{1}{2}$. As in case **A**, $\phi_3 = 0$ is a symmetry, $\phi_3 = \pi$ is redundant, and there is a special allowance for 90 degree rotations, occurring now at $U_y = \frac{1}{2}$. The choice $\alpha = \gamma$ apparently contains no distinct benefit beyond involution **B**. The final observed symmetry occurs for $|U| \equiv \sqrt{U_x^2 + U_y^2} = 1$, i.e. a rhombus with α undetermined. We will label this as case **C**, although it is unknown to us from the prior literature. In this situation, taking $\phi_3 = \alpha(\pm\pi)$ will restore the torus to itself post complex conjugation. For $\alpha = 90°$ this is the square limit of **A**. Taking $\alpha = 60°$ recreates a member from **B** which may be rotated by any multiple of $(2\pi/3)$. In fact, all figures obtained from involution **C** are all *similar* to those of case **B**, although the orientation and scaling are generally distinct. It is true that all final physical formulae are rotationally and scale invariant in terms of conventions on the complex plane. Nevertheless, the basis cycles overlay different sections of the parallelogram within each configuration, and it remains unclear at this stage whether the extant phenomenology are furthered by the third tilting. Analysis of this possibility will be deferred for the time-being. Applying the remaining involutions to each of the three two-tori in all combinations yields only four unique theories, labeled henceforth as $\{\mathbf{AAA}, \mathbf{ABA}, \mathbf{AAB}, \mathbf{ABB}\}$.

$$\begin{aligned}
&\mathbf{A}_3\text{:} \qquad z \quad \rightarrow \bar{z} \qquad\qquad (U_x = 0) \\
&\mathbf{B}_3\text{:} \qquad z \quad \rightarrow \bar{z} \qquad\qquad (U_x = \frac{1}{2}) \\
&\mathbf{C}_3\text{:} \qquad z \quad \rightarrow e^{i\pi\alpha}\bar{z} \qquad (|U| = 1)
\end{aligned} \tag{4.6}$$

In *oriented* string theory e.g. Type **IIA**, the admission of the symmetry $\bar{\sigma}$ of (4.4) on the underlying manifold implies that the combined operator $\Omega\bar{\sigma}$, where Ω represents worldsheet parity, will be a symmetry of the full stringy model. As desired, modulating the spectrum by this '*orientifold* projection' reduces the maximal available bulk supersymmetry to $\mathcal{N} = 1$. However, there are also 'side benefits' at least equally as satisfactory. Namely, this set of identifications causes the generation of a *Ramond–Ramond* charged extended object lying along the invariant plane of the $\bar{\sigma}$ operator (c.f. Section E). This so-called 'orientifold six-plane', π_{O6}, in turn develops a *RR*-

tadpole term which *must* be canceled by the introduction of stacks of $D6$-branes[2], as exhibited in Section F. These stacks of branes are then responsible via string modes tethered at their point of intersection for the appearance of the *massless chiral* matter representations in *gauged multiplets* which we are ultimately seeking, as is to be detailed Section G. Without the $O6$-plane, the right-hand side of (4.36) would vanish, and with it our $D6$-brane stacks, accompanied by the *entire* supersymmetric construction undertaken in the current framework[3].

B. The SUSY Condition on D-Brane Stacks

In preliminary construction of the class of models under consideration, great care has been taken to select a compactification background which can *support* $\mathcal{N} = 1$ supersymmetry. However, this is necessary condition is yet insufficient. One must also certify that the D-Branes wrapped *on* this space are capable of *realizing* supersymmetry in a way which will be manifest within the multiplets arising from strings located at their intersections. The orientation of a stack of D-Branes within T^6 is given as three pair of wrapping numbers (n_i, m_i), with $i = 1, 2, 3$. As suggested in the first diagram of Figure 7, n measures wrapping along the $\hat{x}$ side of the torus, while m counts cycles along the complex structure modulus $\vec{U} = U_x\hat{x} + U_y\hat{y}$. There are three corresponding angles Φ_i defined which measure 'counter-clockwise' from the $+\hat{x}$ axis to the vector $n_i\hat{x} + m_i\vec{U}$. The supersymmetry condition is applied on a per-stack basis, and is enforced by the simple requirement that:

$$\Phi_1 + \Phi_2 + \Phi_3 = 0 \ (\mathbf{mod}\ 2\pi) \tag{4.7}$$

[2]The D-branes are static planes on which open strings terminate when employing the Dirichlet (opposed to Neumann) boundary conditions.

[3]It has alternatively been stated [20] that without the $\Omega\bar{\sigma}$ projection, there are no supersymmetric brane configurations, due to a persistently positive overall tension.

As will be generally the case, all specific discussions in this section take the orientifold $T^6/\mathbb{Z}_4$ context. In this scenario, the complex structure of the first and second 2-Tori is fully determined. This means simply that the prototype parallelogram on which $(n,m)_{1,2}$ are overlaid is not continuously deformable. However, the $\hat{y}$ dimension of the third torus is undetermined and may be freely rescaled by the structure factor U_2. Since just this component of the complex structure will be referenced as a parameter, the special notation of $\mathcal{U}$ is henceforth adopted. A consistent choice of $\mathcal{U}$ for the third toroidal cell is the only inter-stack condition which $SUSY$ imposes on the D-brane configuration. The notion of 'tilting' on the second and third tori will be naturally accommodated in the later discussion, but it remains for now an unnecessary complication. The third intersection angle of a given stack for the AAA involution is then properly defined by the relation

$$\tan(\Phi_3) = \frac{m_3 \mathcal{U}}{n_3} \, . \tag{4.8}$$

Graphically, we can attempt to find the net tangent-angle formed by the first and second intersections, and play this against the expected result from (4.8). In this manner it may be tested whether $SUSY$ is salvaged, and what restrictions are placed on the value of $\mathcal{U}$. The perspective employed will be that each of the $(n,m)_i$ are pre-selected, while $\mathcal{U}$ remains to be set by consistency requirements, as might be convenient for a computer-based search. Discussion here will follow Figure (10). As seen in the first image, the desired value of Φ_3 is such to serve as the explement to $(\Phi_1 + \Phi_2)$. It is convenient to rescale the 2^{nd} triangle to overlay the 1^{st}, as in the second graphic, with

$$n_2' = \sqrt{n_1^2 + m_1^2} \ , \ m_2' = \frac{m_2 n_2'}{n_2} \, . \tag{4.9}$$

The dotted (green) triangle from the last image represents the target angle sum,

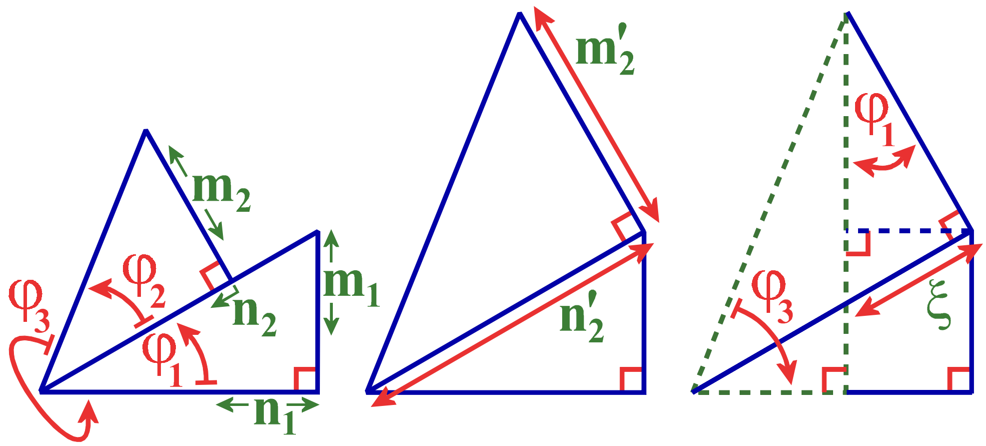

Fig. 10. *Graphical Representation of the SUSY Angle Constraints*

which will be called Φ_{12}. To compute the lengths of its sides, the segment

$$\xi = \frac{m_1 m_2'}{n_1} \tag{4.10}$$

is needed first. We can then write:

$$\tan(\Phi_{12}) = \frac{m_1 + m_2'(\frac{n_1}{n_2'})}{n_1 - \xi(\frac{n_1}{n_2'})} = \frac{m_1 + (\frac{m_2 n_2'}{n_2})(\frac{n_1}{n_2'})}{n_1 - (\frac{m_1}{n_1})(\frac{m_2 n_2'}{n_2})(\frac{n_1}{n_2'})} \tag{4.11}$$

Canceling crossed terms and multiplying through by $(n_2/n_2 \equiv 1)$, this reduces to:

$$\tan(\Phi_{12}) = \frac{m_1 n_2 + n_1 m_2}{n_1 n_2 - m_1 m_2} \tag{4.12}$$

This result is symmetric under the operation $(\Phi_1 \Leftrightarrow \Phi_2)$, consistent with the semantical freedom of our angle designation.

While this construction is visually pleasing, it suffers from at least two shortcomings. Firstly, we have been somewhat incautious with the potentiality of some factors in the calculation to go to zero. Secondly, there is a fundamental property of the tangent function that it is periodic modulo π, rather than 2π, as appears in the $SUSY$ condition of (4.7). This occurs because as each of $\sin(\theta)$ and $\cos(\theta)$ take on a negative sign with the advance ($\theta \rightarrow \theta \pm \pi$), the ratio $\tan(\theta)$ remains constant. Equivalently, multiplication of each the numerator and denominator of a tangent ration by a negatively signed factor will reflect the quadrant to which the associated angle vector refers. In order to unequivocally determine the referenced angle, it is necessary to know at least the numerator or denominator's sign in addition to the tangent value. By choosing to flip the signs of $(n,m)_i$, the diagrams of Figure (10) can suffer a severe distortion. In conjunction with the through multiplication of n_2, this may leave it unclear at the outset that signs have been treated with sufficient delicacy[4]. An alternate and more elegant derivation can help to rescue confidence in (4.12).

We will consider that each of $(n,m)_1$ and $(n,m)_2$ establish a vector in the plane. The matrix transformation which rotates such a vector through an angle Φ_1 is given by:

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \frac{1}{\sqrt{n_1^2+m_1^2}} \otimes \begin{pmatrix} n_1 & -m_1 \\ m_1 & n_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{4.13}$$

Application of the formula (4.13) to the 2^{nd} set of wrapping numbers again effectively adds the angles Φ_1 and Φ_2, and yields barred coordinates within the ray lying along this sum.

$$\begin{pmatrix} \bar{n_2} \\ \bar{m_2} \end{pmatrix} = \frac{1}{\sqrt{n_1^2+m_1^2}} \otimes \begin{pmatrix} n_1 n_2 - m_1 m_2 \\ m_1 n_2 + n_1 m_2 \end{pmatrix} \tag{4.14}$$

Associating

$$\tan(\Phi_{12}) = \frac{\bar{m_2}}{\bar{n_2}}\,, \tag{4.15}$$

it is clear that (4.12) is reproduced identically. Cancellation of the positive-definite

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[4]In fact, making sign assumptions *consistent* with the diagram figure *are* sufficient to protect the *dependent* derivation under all deformations.

radical $\sqrt{n_1^2 + m_1^2}$ is free from ambiguity.

We note that the target ray for the 3^{rd} wrapping, such to achieve the $\sum_{i=1}^{3} \Phi_i = 0$ ($\mathbf{mod}$ 2π) rule, is marked always by flipping the numerator $\bar{m}_2 \to -\bar{m}_2$ in (4.15). It will be convenient at this point then to make two definitions.

$$\begin{aligned} \Gamma &\equiv -(m_1 n_2 + n_1 m_2) \\ \Delta &\equiv +(n_1 n_2 - m_1 m_2) \end{aligned} \tag{4.16}$$

It can thus be stated that

$$\frac{m_3 \mathcal{U}}{n_3} = \frac{\Gamma}{\Delta}\,, \tag{4.17}$$

employing (4.8,4.12) with the recent definitions. We stress that embedded within this equivalency is *also* an understanding that the related numerators (denominators) should share a common sign. A sequence of rules in the format of a logical flow can now be read off of the previous relationship, as exhibited in Table II. We have allowed either sign in general, but neither the values of zero nor infinity for $\mathcal{U}$, the $\hat{y}$ scaling factor (complex structure) on the third 2-torus. This prevents both collapse and an infinite extent for the physical notion of the compact surface element. Although we have elected here not to *manually* restrict U to the upper half-plane this condition will reappear *dynamically* in Section F. The condition of an indeterminate result for (4.17) does not occur. To see this, define two vectors

$$\vec{a} \equiv n_1 \hat{x} + m_1 \hat{y}\ ,\ \vec{b} \equiv n_2 \hat{x} - m_2 \hat{y} \tag{4.18}$$

embedded in a three-space. Then,

$$\Gamma = (\vec{a} \times \vec{b}) \cdot \hat{z}\ ,\ \Delta = (\vec{a} \cdot \vec{b}) \tag{4.19}$$

These can ever simultaneously vanish only if $\vec{a} = 0$ or $\vec{b} = 0$, which are disallowed as

Table II. Rules for the Preservation of Supersymmetry in terms of Wrapping Numbers

1. If Γ equals zero
   - m_3 must be zero
   - $(n_3\Delta)$ must be positive
   - $\mathcal{U}$ is undetermined
2. If Δ equals zero
   - n_3 must be zero
   - $\mathcal{U}$ must carry the sign of $(m_3\Gamma)$
3. Otherwise
   - $(n_3 m_3)$ cannot be zero
   - $(n_3\Delta)$ must be positive
   - $\mathcal{U} = \frac{n_3\Gamma}{m_3\Delta}$ is fixed

null winding configurations. This conclusion is consistent with the intuition of Γ/Δ as the tangent of a physical angle.

Having dealt with the $SUSY$ condition for the model **AAA**, it is time to consider the tilted involutions. If the second torus is of type **B**, (4.7) is replaced by the condition:

$$\Phi_1 + \Phi_2 + \Phi_3 = \pi/4 \ (\mathbf{mod}\ 2\pi) \tag{4.20}$$

The revised angle is a direct consequence of the $\pi/4$ rotation undertaken by the $O6$-plane in this construction, as discussed in Section E. It will be convenient to absorb these 45 degrees into a redefinition of $(n,m)_2$ which is *counter* rotated by $-\pi/4$.

$$\begin{pmatrix} \tilde{n}_2 \\ \tilde{m}_2 \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \otimes \begin{pmatrix} +n_2 + m_2 \\ -n_2 + m_2 \end{pmatrix} \tag{4.21}$$

Replacing $(n,m)_2$ with $(\tilde{n},\tilde{m})_2$, all the previous analysis goes through smoothly, with the angles once again summing to zero in the effective wrapping number frame. In terms of the tangent definition with respect to the original coefficients,

$$\begin{aligned} \frac{\Gamma}{\Delta} &\Rightarrow \frac{-(m_1 n_2 + m_1 m_2 - n_1 n_2 + n_1 m_2)}{+(n_1 n_2 + n_1 m_2 + m_1 n_2 - m_1 m_2)} \\ &= \frac{\Delta + \Gamma}{\Delta - \Gamma} \equiv \frac{\tilde{\Gamma}}{\tilde{\Delta}} \end{aligned} \tag{4.22}$$

It is clear that indeterminacy of the transformed ratio is likewise excluded, as it would require the previously disallowed scenario $\Gamma = \Delta = 0$. Although the 'natural' split definitions of $\tilde{\Gamma}$ and $\tilde{\Delta}$ could be argued to each carry a factor of $1/\sqrt{2}$, we will forgo this for the simpler cross relation of quotients implied by (4.22).

$$\tilde{\Gamma}_B \equiv (\Delta + \Gamma)_A \ , \ \tilde{\Delta}_B \equiv (\Delta - \Gamma)_A \tag{4.23}$$

We will advocate the position that the unwieldy $(n,m)_{1,2}$ be replaced whenever possible by the more compact factors of $\tilde{\Gamma}$ and $\tilde{\Delta}$. The question of what limitations are imposed by ultimately demanding an explicit representation of $(\tilde{\Gamma},\tilde{\Delta})$ in terms of fundamental co-prime wrapping numbers will be saved for later.

Tilting of the third torus is handled similarly. Because the vector $\vec{U}$ now contains a lateral element, the m wrapping coefficients additionally supply a displacement in the $\hat{x}$ direction of one-half unit per cycle (c.f. Eq. 4.6). The third intersection angle is thus replaced by

$$\tan(\Phi_3) \Rightarrow \frac{m_3 \mathcal{U}}{n_3 + \frac{1}{2} m_3} \equiv \frac{\tilde{m}_3 \mathcal{U}}{\tilde{n}_3} \, , \tag{4.24}$$

recalling that $\mathcal{U}$ still refers to just the $\hat{y}$ component of the full complex structure. When there is no ambiguity, the subscript will be dropped from $(\tilde{n}_3, \tilde{m}_3)$, expressing them simply as $(\tilde{n}, \tilde{m})$. The explicit split definitions for these effective wrappings are:

$$\tilde{m}_B \equiv m_A \ , \ \tilde{n}_B \equiv (n + \frac{m}{2})_A \tag{4.25}$$

The generalized $SUSY$ requirement is thus restated

$$\frac{\tilde{m}\mathcal{U}}{\tilde{n}} = \frac{\tilde{\Gamma}}{\tilde{\Delta}} \, , \tag{4.26}$$

understanding that the definitions of $(\tilde{n}, \tilde{m})_{2,3}$ are to be employed as in (4.23,4.25) *only* whenever the corresponding torus represents a **B**-type involution. The omission of a tilde-notation in any formula will imply reversion to the definitions of (4.16), irrespective of any tilting.

C. Fundamental Bulk Basis Cycles

Again the context of current discussion will be $T^6/\mathbb{Z}_4$. A complete basis set of 16 linearly independent $\mathbb{Z}_4$ invariant 3-cycles has been developed in reference [20]. These are partitioned into two sets of 8, with a barred notation marking dependence on the 'm' (as opposed to 'n') wrapping of the 3^{rd} torus, all else being equal. There is a further subdivision, as just two of these pair $(\rho, \bar{\rho})_{1,2}$ descend from the ambient toroidal 'bulk' space, being simple combinations of the fundamental wrappings on T^6. The remaining 6 pair $(\varepsilon, \bar{\varepsilon})_{i=1...6}$ are 'exceptional cycles' arising from the $\mathbb{Z}_2$ 'twisted' orbifold sector. Their discussion will resume in the following section. The projections of a given wrapping onto each of the bulk basis members are a function of the $(n, m)_i$ selected for a given stack. These coefficients of the bulk cycles take on a simple form in our current language.

$$\pi^{\mathrm{b}} = [n\Delta]\rho_1 + [-n\Gamma]\rho_2 + [m\Delta]\bar{\rho}_1 + [-m\Gamma]\bar{\rho}_2 \tag{4.27}$$

Notice that this formula is static across all four involutions. We will need also the image π' under the orientifold mirror $\Omega\bar{\sigma}$. The transformation is enacted via 4×4 matrix Σ, *distinct* per each case of tilting.

$$\pi' \equiv \Sigma \times \pi \tag{4.28}$$

It is convenient to first define a pair of 2-dimensional square matrices:

$$\alpha_2 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;,\; \beta_2 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.29}$$

Then:

$$\Sigma \;=\; \underset{\textbf{AAA}}{\begin{pmatrix} \alpha & \emptyset \\ \emptyset & -\alpha \end{pmatrix}} \;;\; \underset{\textbf{ABA}}{\begin{pmatrix} \beta & \emptyset \\ \emptyset & -\beta \end{pmatrix}} \;;\; \underset{\textbf{AAB}}{\begin{pmatrix} \alpha & \alpha \\ \emptyset & -\alpha \end{pmatrix}} \;;\; \underset{\textbf{ABB}}{\begin{pmatrix} \beta & \beta \\ \emptyset & -\beta \end{pmatrix}} \tag{4.30}$$

Since the matrices α_2 and β_2 square to unity, a repeated application of Σ maps all cycles to themselves as expected.

D. Exceptional Cycles and Fractional Branes

The so-called 'fractional' branes are pairings of a fundamental cycle with one of the exceptional classes. A factor of 1/2 is pre-pended to the fundamental cycle whenever such a combination is considered. Exceptional cycles ever appear only in such unions, rather than as isolated manifestations of the D-brane wrapping state. Although the description of fractional branes is demanded for completeness, and they can certainly be phenomenologically indicated for the purposes of model building, in no situation is their *use* mandatory. On the other hand, severe restrictions do exist on when and how we *may* elect to employ the exceptional sector, as provided in Table III [5]. These

Table III. Allowed Exceptional Cycles

	n_1 odd, m_1 odd	n_1 odd, m_1 even	n_1 even, m_1 odd
n_2 odd, m_2 odd	**(Case 0)**	$\varepsilon_3, \varepsilon_4$ $\varepsilon_5, \varepsilon_6$ **(Case 2)**	$\varepsilon_3, \varepsilon_4$ $\varepsilon_5, \varepsilon_6$ **(Case 2)**
n_2 odd, m_2 even	$\varepsilon_1, \varepsilon_2$ $\varepsilon_5, \varepsilon_6$ **(Case 1)**	$\varepsilon_1, \varepsilon_3, \varepsilon_5$ $\varepsilon_1, \varepsilon_4, \varepsilon_6$ $\varepsilon_2, \varepsilon_3, \varepsilon_6$ $\varepsilon_2, \varepsilon_4, \varepsilon_5$ **(Case 3)**	$\varepsilon_1, \varepsilon_3, \varepsilon_6$ $\varepsilon_1, \varepsilon_4, \varepsilon_5$ $\varepsilon_2, \varepsilon_3, \varepsilon_5$ $\varepsilon_2, \varepsilon_4, \varepsilon_6$ **(Case 4)**
n_2 even, m_2 odd	$\varepsilon_1, \varepsilon_2$ $\varepsilon_5, \varepsilon_6$ **(Case 1)**	$\varepsilon_1, \varepsilon_3, \varepsilon_6$ $\varepsilon_1, \varepsilon_4, \varepsilon_5$ $\varepsilon_2, \varepsilon_3, \varepsilon_5$ $\varepsilon_2, \varepsilon_4, \varepsilon_6$ **(Case 4)**	$\varepsilon_1, \varepsilon_3, \varepsilon_5$ $\varepsilon_1, \varepsilon_4, \varepsilon_6$ $\varepsilon_2, \varepsilon_3, \varepsilon_6$ $\varepsilon_2, \varepsilon_4, \varepsilon_5$ **(Case 3)**

restrictions emerge from the consideration that only those exceptional cycles may contribute which are intersected by the flat D-brane. Notice that Case 1 versus Case 2 represents an inversion of the first and second tori. The distinction between Cases 3 and 4 is marked by the interchange ($\varepsilon_5 \Leftrightarrow \varepsilon_6$). Depending on the case in play,

[5]Taken with slight modification from its appearance as Table 3 of reference [20].

as selected by the wrapping 'parity' on the first and second tori, there are 2, 4, or 0 choices for the set of non-vanishing exceptional cycle elements. A 6-dimensional vector $\vec{\omega}$ is defined for each given set, bearing a (freely) signed coefficient $(\pm 1)^6$ for the two (or three) listed basis elements, and all other entries null. The exceptional cycle is then fully determined, with the *same* vector $\vec{\omega}$ providing signs for *both* of the six $(\varepsilon, \bar{\varepsilon})$, while each is *separately* mitigated in scale by wrapping numbers on the 3^{rd} torus.

$$\pi^{\mathrm{e}} = n(\vec{\omega} \cdot \vec{\varepsilon}) + m(\vec{\omega} \cdot \vec{\bar{\varepsilon}}) \tag{4.31}$$

The net fractional homology is the halved-sum of both bulk (c.f. Eq. 4.27) and exceptional contributions.

$$\pi^{\mathrm{f}} = \frac{1}{2}\pi^{\mathrm{b}} + \frac{1}{2}\pi^{\mathrm{e}} \tag{4.32}$$

While Table III represents the natural language of derivation for its constraints, the provided framework is somewhat bulky, and inconvenient for computational application. The reader may readily confirm equivalency of the reduced rule set appearing in Table IV. Following our injunction to favor (Γ, Δ) of (4.16) over the $(n, m)_{1,2}$, we can also (nearly) restate the allowed exceptional cycles in this language (Table V). As was hinted previously, the ability to distinguish between Cases $(1, 2)$ requires a detailed knowledge of wrapping on the 1^{st} and 2^{nd} tori. However, the factors (Γ, Δ) treat those wrapping numbers in full symmetry. Regardless, this will be the *least* significant distinction between the 5 possible classifications.

As with the bulk cycles, it will be necessary to know the orientifold $\Omega\bar{\sigma}$ mapping of the exceptions. Just as in (4.28), $\pi' \equiv \Sigma \times \pi$, except Σ is now a 12×12 matrix, where the upper six rows operate on ε_i, and the lower on $\bar{\varepsilon}_i$. The block-composition

[6] This relative sign freedom is in correspondence to the turning on of a discrete Wilson line along an internal D-brane direction at the orbifold point.

Table IV. Rules for Selection of the Allowed Exceptional Class

1. If $(n_1 + m_1 + n_2 + m_2)$ is Even
   (a) If $(n_1 + n_2)$ is Even
      i. If $(n_1 \times m_1)$ is Even : **Case 3**
      ii. Otherwise : **Case 0**
   (b) Otherwise : **Case 4**
2. Otherwise
   (a) If $(n_1 \times m_1)$ is Even : **Case 2**
   (b) Otherwise : **Case 1**

of Σ from (4.30) remains intact across each tilted involution, so long as we update (4.29) with 6-dimensional matrices.

$$\alpha_6 \equiv \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} , \quad \beta_6 \equiv \begin{pmatrix} -1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix} \tag{4.33}$$

Unmarked off-diagonal elements are zero. As before, $\Sigma^2 \equiv \mathbb{1}$.

Table V. Allowed Exceptional Cases in Terms of (Γ, Δ)

	Γ odd	Γ even
Δ odd	**Case 1** *or* **2**	**Case 3**
Δ even	**Case 4**	**Case 0**

E. The O6 Planes

Under the action of the symmetries imposed on the torus T^6, there are certain sets of fixed points which become re-associated into themselves. Since the $\mathbb{Z}_4$ action Θ represents a rotation in the complex plane by $e^{\frac{i2\pi}{4}}$, or '90 degrees counter-clockwise', on the first two tori T^{2}[7], it may be applied up to three times (Θ^3) before becoming degenerate. From each of operations $\Omega\bar{\sigma}$, $\Omega\bar{\sigma}\Theta$, $\Omega\bar{\sigma}\Theta^2$, and $\Omega\bar{\sigma}\Theta^3$, there is defined an invariant plane which bisects each 2-torus in a manner identical to the cross-cutting of the D-branes themselves. In fact, the orientation these fixed point sets can be fully described in terms of *just* the four bulk wrapping basis cycles. The sum of the four listed contributions will be called the 'orientifold six-plane', π_{O6}. Under each allowed tilting, as induced by distinct 'anti-holomorphic involutions' $\bar{\sigma}\colon z \to e^{i\phi_i}\bar{z}$, there is a

[7] Θ operates on the 3^{rd} torus as $e^{-i\pi}$, balancing its net effect to unity. In this context then, $\Theta^2 \equiv \mathbb{1}$.

unique resulting $O6$-plane.

$$\pi_{O6} = \left\{ \begin{array}{cccc|c} \rho_1 & \rho_2 & \bar{\rho}_1 & \bar{\rho}_2 & \\ \hline 4 & 0 & 0 & -2 & \mathbf{AAA} \\ 2 & 2 & 2 & -2 & \mathbf{ABA} \\ 2 & 1 & 0 & -2 & \mathbf{AAB} \\ 0 & 2 & 2 & -2 & \mathbf{ABB} \end{array} \right. \tag{4.34}$$

F. Tadpole Constraints

Once the orientifold projection and associated invariant planes are incorporated into our construction, they will necessarily carry a non-vanishing induced Ramond-Ramond charge. These structures are manifest in the string theory as the appearance of a RR-tadpole. Suppression of this undesirable factor is in fact the underlying *natural* and *essential* motivation for the introduction of D-brane stacks wrapping the compact space, and with them gauged chiral matter multiplets as are expected in the standard model and GUT extensions. It also provides one of strongest guiding constraints as to how the allocation of these stacks may be consistently realized.

The full constraint is written concisely as a sum over all stacks a, with N_a the number of D-branes in the a^{th} stack.

$$\sum_a N_a(\pi_a + \pi'_a) = 4\pi_{O6} \tag{4.35}$$

However, in this language the underlying similarities between tilted variants are somewhat masked. Note that (4.35) actually represents 16 relationships, as it must be satisfied individually for each of the basis elements of the cycles π. We shall choose instead to expand the constraints in terms of these sixteen coefficient sets. Since the $O6$-plane sources associate to bulk cycles only, the *four* $(\rho, \bar{\rho})_{1,2}$ are conveniently grouped together, and will be addressed first, using (4.27,4.28,4.34). Taking each involution in turn, it is seen that there are only ever *two* independent bulk tadpole conditions, both of which exhibit a pleasing unified form.

$$\begin{aligned} \sum_a \left(\{2\}^{\mathrm{b}} N \tilde{n} \tilde{\Delta}\right)_a &= \Pi_n \\ \sum_a \left(\{2\}^{\mathrm{b}} N \tilde{m} \tilde{\Gamma}\right)_a &= \Pi_m \end{aligned} \tag{4.36}$$

The term $\{2\}^{\mathrm{b}}$ is a reminder that stacks which wrap *only* in the bulk, as opposed to the fractional branes, contribute at *twice* the rate to saturation of the tadpole. The $\Pi_{(n,m)}$ are (distinct) constant positive numbers in each tilted configuration.

$$\begin{array}{ccc|c|c|c} & & \mathbf{AAA} & \mathbf{ABA} & \mathbf{AAB} & \mathbf{ABB} \\ \hline \Pi_n & = & 16 & 16 & 8 & 8 \\ \Pi_m & = & 8 & 16 & 8 & 16 \end{array} \tag{4.37}$$

For the remaining 12 tadpole constraints on exceptional cycles (c.f. Eqs. 4.31,4.28,4.33) the right-hand side of (4.35) is zero. As before, only one half of the possible constraints are non-trivially applied. There is an *apparent* disparity in how these six equations are written, based on whether or not the *second* torus is tilted.

$$\sum_a N_a (\tilde{n}\vec{\omega})_a = 0 \qquad (\mathbf{AAA}, \mathbf{AAB}) \tag{4.38}$$

$$\sum_a N_a \left(\tilde{m}\omega_1, \tilde{m}\omega_2, \tilde{n}\omega_3, \tilde{n}\omega_4, \tilde{n}\frac{(\omega_5+\omega_6)}{2}, \tilde{m}\frac{(\omega_5-\omega_6)}{2}\right)_a = 0 \qquad (\mathbf{ABA}, \mathbf{ABB}) \tag{4.39}$$

We will now begin to cross-reference the bulk tadpole constraint of (4.36) with the rules on supersymmetry preservation from Table II[8]. First it is noted that under *all* scenarios, the quantity $\tilde{n}\tilde{\Delta}$ is greater than or equal to zero, by $SUSY$. Next, from

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[8]Although this table was constructed prior to introduction of the 'tilde' notation, it *does indeed* refer to the rotated effective wrapping numbers.

the tadpole constraint, we observe that at least *some* stacks must take a positive value for $\tilde{m}\tilde{\Gamma}$. Whenever this is the case, the $SUSY$ rules say that $\mathcal{U}$, the (imaginary part of the) complex structure on the third torus must also be positive. But, since *every* stack must share just a *single* physical value for $\mathcal{U}$, this value must therefore always be positive. The converse is also true, in that a positive $\mathcal{U}$ cannot ever allow $\tilde{m}\tilde{\Gamma}$ to be negative. Therefore, *all* contributions to the two bulk tadpole terms are strictly positive semi-definite. Furthermore, since $(\tilde{n}\tilde{\Delta})$ and $(\tilde{m}\tilde{\Gamma})$ must never simultaneously vanish[9], *all* stacks must contribute to at least one of the terms. These conditions are highly confining, and disallow the prospect of an indefinite well of D-brane stacks or large wrapping numbers from which to fashion any desired result. The least restrictive involution is the ABA configuration, in the sense that its tadpole allowances from (4.37) are the greatest.

Table II is then be updated by Table VI, reflecting this new knowledge. Please note that Table VI does *not* subsume (4.36), but instead they must be taken in *conjunction*.

Finally, we summarize a number of generic statements which can now be made after combination of the $SUSY$ and RR-tadpole constraints.

1. $\mathcal{U}$ must be a positive number.

2. All stacks make a positive-definite contribution to at least one bulk tadpole.

3. Only stacks which hit both tadpoles fix $\mathcal{U}$ numerically.

[9]Note however that when $(\tilde{n}\tilde{\Delta})$ is zero, *both* of its factors vanish together, and likewise for $(\tilde{m}\tilde{\Gamma})$.

Table VI. The Combined $SUSY$ plus RR-tadpole Wrapping Selection Rules

1. If $(\tilde{n}\tilde{\Delta} < 0)$ *or* $(\tilde{m}\tilde{\Gamma} < 0)$
   - The construction Fails
2. If $(\tilde{n}\tilde{\Gamma})$ equals zero
   - $(\tilde{m}\tilde{\Delta})$ must be zero
   - $\mathcal{U}$ is positive
3. Otherwise
   - $(\tilde{m}\tilde{\Delta})$ cannot be zero
   - $\mathcal{U} = \frac{\tilde{n}\tilde{\Gamma}}{\tilde{m}\tilde{\Delta}}$ is fixed

G. Brane Intersections and Gauged Symmetry Groups

The static position of a D-brane within a compactified space has the T-dual interpretation of Wilson-line type phases accrued by Chan-Patton charges at the ends of open strings wrapping the non-trivial topology. The cyclic nature of the trace factor in tree-level scattering for *open oriented* strings with N CP degrees of freedom guarantees that such constructions generally support a globalized $U(N)$ symmetry[1]. There is the generic expectation in string theory that global world-sheet symmetries will be dynamically gauged in space-time by creation operator modes which carry the internal indices. However, this symmetry is broken if the N Wilson phases are

[1]There are also mechanisms for achieving Orthogonal (unoriented) and Symplectic gauge groups.

non-degenerate, or in the other picture, if the N corresponding D-branes take on distinct positions. Alternatively, the gauge symmetry may be viewed as being gauged by the N^2 massless modes associated with the N redundant string attachment faces of the stacked D-branes. It is then 'Higgsed' if the member elements diverge from each other, thus creating *massive* modes corresponding to a non-zero energy of stretching.

The presence of magnetic fluxes from such gaugings corresponds in a dualized picture to a tilting of the related D-brane. There will thus generally be intersections between any various such stacks, at which (string scale) massless modes are expected to materialize. The resulting particle spectrum will carry bi-fundamental representations from the two stacks on which its endpoints affix. The redundancy, or number of generations in the usual particle language, of gauge multiplets produced by stacks of intersecting D-branes is simply determined by the number of times which the stacks bisect each other while wrapping the fundamental toroidal section. Some intuition for the calculation of this 'topological intersection number' can be gained by the elementary graphical demonstration of Figure (11). However, care must be taken in that we will ultimately wish to deal with the orbifolded quotient space, and not the ambient torus T^6 itself. In this picture, two wrapping vectors (large blue & red arrows) we can call $(\vec{a}, \vec{b})$ overlay the (square) toroidal lattice. In the upper left-hand corner all crossings of each vector are overlaid into a single cell so that their intersections (green dots) may be counted. Note that we could instead choose to view one of the two vectors as in the expanded lattice while it is repeatedly crossed by the dotted arrows of the opposing color. However, the counting of either set of dotted lines (being careful to not apply the edges in duplicate) is just equivalent to the number of internal lattice points framed by the parallelogram of the wrapping vectors. To verify, note that each lattice point must by definition form the origin (and destination) of a translated

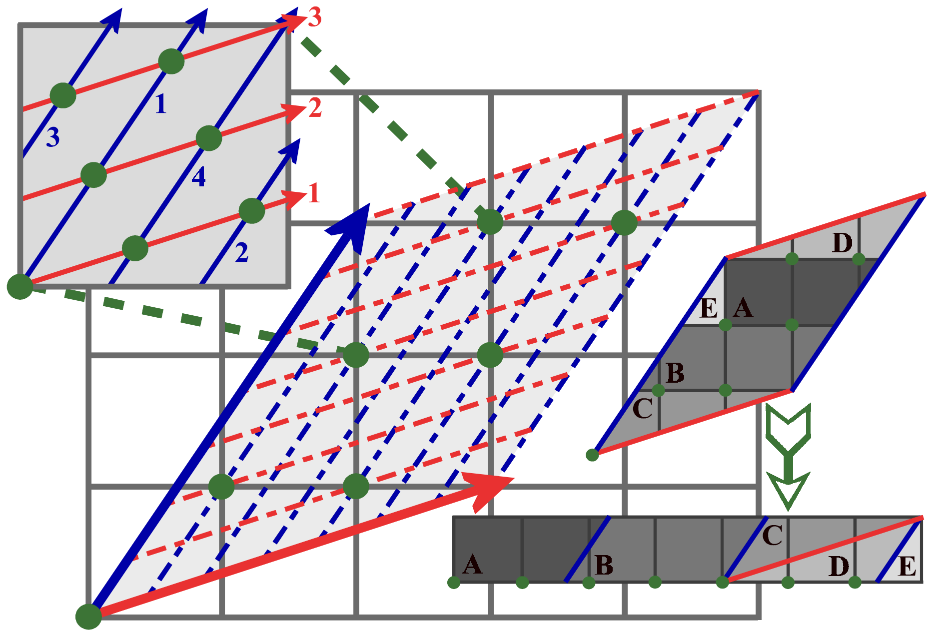

Fig. 11. *Graphical Representation of the Topological Intersection Number*

version of the wrapping vectors $\vec{a}$ *and* $\vec{b}$. The 'co-prime'[2] wrapping restriction against multiple windings ensures that any such point will be crossed no more than once. The right-hand side of Figure 11 demonstrates via rearrangement of the parallelogram[3] that the desired count of contained lattice nodes (green dots) is further equivalent to the enclosed *area*. There is then a convenient vector expression for the number of resulting families in terms of a cross product.

$$M_{ab} = (\vec{a} \times \vec{b}) \cdot \hat{z} = (n_a m_b - m_a n_b) \tag{4.40}$$

As expected for consistency, this is purely integral. Within the definition of sign convention as in (4.40), the exchange $(a \Leftrightarrow b)$ is antisymmetric. Negative values of

[2]Co-prime wrapping numbers (n, m) are selected such that they have no *common integral* divisors beside ± 1.

[3]As is coded alphabetically and in gray-scale.

M_{ab} imply conjugate representations. Although this exercise was undertaken for a square lattice, the result is valid for any conformal distortion, i.e. alternate value of the complex structure, *as long as* the wrapping numbers (n, m) are the pure *count* of windings *along* the skewed axes. In other words, particle multiplicities are expected *not* to refer to the effective winding coefficients $(\tilde{n}, \tilde{m})$. However, appearance of the trigonometric function $\sin(\Phi_{ab})$ in this construction should hearken back to the $SUSY$ constraint of (4.7), and provide some intuition for the recurrence of the familiar quantities (Γ, Δ).

The specific results for the $d = 4$ chiral gauged multiplets in $T^6/\mathbb{Z}_4$ are provided in term of the previously described basis cycles and orientifold plane in Table VII [4]. We note that all gauge multiplets derived from this and related models are *guaranteed* to be free of non-Abelian anomalies, which forms then a valuable consistency check on the resultant spectra. The circle-product '$\circ$' notation here signifies application of the

Table VII. Chiral Spectrum in $d = 4$

Representation	Multiplicity
$[\mathbf{A_a}]_L$	$\frac{1}{2}\left(\pi'_a \circ \pi_a + \pi_{\mathrm{O6}} \circ \pi_a\right)$
$[\mathbf{S_a}]_L$	$\frac{1}{2}\left(\pi'_a \circ \pi_a - \pi_{\mathrm{O6}} \circ \pi_a\right)$
$[(\overline{\mathbf{N}}_\mathbf{a}, \mathbf{N_b})]_L$	$\pi_a \circ \pi_b$
$[(\mathbf{N_a}, \mathbf{N_b})]_L$	$\pi'_a \circ \pi_b$

topological intersection between cycles. The basis cycles, which we will collectively name $\pi_i \equiv (\rho_i, \varepsilon_i)$, were defined such to be orthogonal under intersection. There is a non-vanishing product *only* with the barred-partner of each cycle, normalized to

[4] Appears originally as Table 1 of reference [20].

magnitude 2. Commuting the product yields a Grassman-like flip of sign.

$$\bar{\pi}_i^a \circ \pi_j^b = 2\delta_{ij}\epsilon^{ab} \tag{4.41}$$

It is now possible to expand the intersections from Table VII into the more tangible language of wrapping numbers and exceptional vector elements, using (4.16,4.27,4.31 & 4.32).

$$\begin{aligned} \pi_a \circ \pi_b \quad = \quad & 2\left\{\tfrac{1}{2}\right\}_a^{\mathrm{f}}\left\{\tfrac{1}{2}\right\}_b^{\mathrm{f}} \times [-(n\Delta)_a(m\Delta)_b + (m\Gamma)_a(n\Gamma)_b \\ & \qquad -(n\Gamma)_a(m\Gamma)_b + (m\Delta)_a(n\Delta)_b] \\ & +\tfrac{1}{2}\,\vec{\omega}_a \cdot \vec{\omega}_b \times [-n_a m_b + m_a n_b] \end{aligned} \tag{4.42}$$

$$\begin{aligned} \pi_a' \circ \pi_b \quad = \quad & 2\left\{\tfrac{1}{2}\right\}_a^{\mathrm{f}}\left\{\tfrac{1}{2}\right\}_b^{\mathrm{f}} \times [-(n\Delta)_a(m\Delta)_b + (m\Gamma)_a(n\Gamma)_b \\ & \qquad +(n\Gamma)_a(m\Gamma)_b - (m\Delta)_a(n\Delta)_b] \\ & +\tfrac{1}{2}\,\vec{\omega}_a \cdot \vec{\omega}_b \times [-n_a m_b - m_a n_b] \end{aligned} \tag{4.43}$$

In each expression the first square brackets enclose the contribution of the bulk cycles while the second set describe the exceptions, if any. In that case of fractional cycles, the $\{\frac{1}{2}\}^{\mathrm{f}}$ coefficients must also be employed. The orientifold six-planes ostensibly require special treatment, in that each tilted involution offers a unique value (c.f. Eq. 4.34).

$$\pi_{O6} \circ \pi_a = \left\{\tfrac{1}{2}\right\}_a^{\mathrm{f}} \times \left\{ \begin{array}{cccc|c} -8m\Delta & & & +4n\Gamma & \mathbf{AAA} \\ \hline -4m\Delta & +4m\Gamma & +4n\Delta & +4n\Gamma & \mathbf{ABA} \\ \hline -4m\Delta & +2m\Gamma & & +4n\Gamma & \mathbf{AAB} \\ \hline & +4m\Gamma & +4n\Delta & +4n\Gamma & \mathbf{ABB} \end{array} \right. \tag{4.44}$$

However, under the transformations (4.23,4.25), and using the constants of (4.36), a unified form is realized.

$$\pi_{O6} \circ \pi_a = 4 \left\{ \tfrac{1}{2} \right\}_a^{\mathrm{f}} \times \left[\tilde{n}\tilde{\Gamma} - \tilde{m}\tilde{\Delta} \left(\frac{\Pi_n}{\Pi_m} \right) \right] \tag{4.45}$$

H. Searching for Flipped $SU(5)$

Since the Grand Unified Theory of 'Flipped $SU(5)$' holds for us a special appeal, it will be made the target in this section as an example of model construction. Seeking out the gauge group $SU(5) \times U(1)_X$ then recommends the use of stacks with either five or just a single element. The chiral matter representations in this scenario are

$$\left\{ \mathbf{10}\,,\, \bar{\mathbf{5}}^{\mathrm{F}}\,,\, \mathbf{1}^{\mathrm{F}} \right\}_L \,, \tag{4.46}$$

times three generations. The superscript 'F' is in place to emphasize that the 'flippings' ($u_L^c \Leftrightarrow d_L^c$) and ($\nu_l^c \Leftrightarrow e_L^c$) restructure the charge content of our five-bar and singlet representations as compared to Georgi-Glashow theory. The flipped-type $U(1)_X$ (non-anomalous) multiplets $\bar{\mathbf{5}}^{\mathrm{F}}$ come from the the $[(\overline{\mathbf{N}}_{\mathbf{a}}, \mathbf{N}_{\mathbf{b}})]$ elements of Table VII, while $[(\mathbf{N}_{\mathbf{a}}, \mathbf{N}_{\mathbf{b}})]$ produces standard $SU(5)$ content from strings stretched between five- and single-member stacks. For the singlets this assignation is reversed, with $\mathbf{1}^{\mathrm{F}}$ arising out of either the $[(\mathbf{N}_{\mathbf{a}}, \mathbf{N}_{\mathbf{b}})]$ stretching between isolated D-branes *or* the symmetric self-attachment of a single brane. There is no such distinction for the $\mathbf{10}$. The Georgi-Glashow five-plets are not without use to us however, as we take a pair $(\mathbf{5}, \bar{\mathbf{5}})$ to serve as Standard-Model Higgs. It is also desirable to find a pair $((\mathbf{10}, \overline{\mathbf{10}}))$ with which to break the *GUT to* the Standard Model[5]. Although mechanisms exist for pushing 'extraneous' content to an unobservable string-scale mass, we shall nev-

[5]That this is possible via antisymmetric representations (which now house an electrically *neutral* member) rather than the usual adjoint Higgs is an advertised feature of Flipped $SU(5)$.

ertheless endeavor where possible to achieve the *minimal* representation, free of any (symmetric) **15**'s and enlarged matter or Higgs sectors. As an aside, each family in (4.46) exactly represents the decomposition of a **16** (spinor) of $SO(10)$.

We shall resist for now the complication of tilting until the prototype scenario of **AAA** has been laid out in full. It is sensible to start first with the most broad yet inescapable of our phenomenological demands. This will be simply that the number of antisymmetric **10**-plets is *nonzero*. The first critical observation which can be made in this context is that contributions to *either* of the 'self-intersection' type multiplicities *require* that a stack participate in *both* bulk tadpole terms of (4.36), and thus that it sets a hard value for $\mathcal{U}$. To see this, note that every term from both of (4.43,4.45) contains at least one member from each of the pairs $\tilde{n}\tilde{\Delta})_a$ and $\tilde{m}\tilde{\Gamma})_a$ (for $a = b$), so that the vanishing of either[6] kills the entire result. The *least* possible value which can be applied to $(Nm\Gamma)_a$ for our stack of five D-branes is just 5. In the current **AAA** scenario this is the *only* choice available, and in fact it is *necessary* that exceptional cycles be include so that the $\{2\}^{\mathbf{b}}$ factor not oversaturate the 2^{nd} tadpole. Since the value of Π_n is here 16, not just 8, we can in principle take any of $n\Delta = (1, 2, 3)$. However, the exceptional cycle tadpoles of (4.38) must also be taken into account.

It is important to notice here that only contributors to the 1^{st} bulk tadpole "$n\Delta$" have any effect on the exceptional tadpole. Factors of both sign clearly appear in this zero-sum constraint. However, it is certainly not a free-for-all, in that even by assuming the *minimal* case of $\delta = \pm 1$ a contribution of magnitude $|N_a n_a|$ is *at least* matched by the bulk tadpole. Therefore, if the exceptional tadpole is *overextended* by more than *half* the total allocated by $\Pi_n \Rightarrow 16$, there will not be enough 'gas' left

[6]Recall consistency enforces that both related members mush go to zero together, if at all.

to get back home[7].

$$N_a n_a \leq 16 \qquad (\mathbf{AAA}) \tag{4.47}$$

This is enough to already to restrict our consideration to just

$$(n\Delta)_5 = 1 \qquad ; \qquad (m\Gamma)_5 = 1\,. \qquad (\mathbf{AAA}) \tag{4.48}$$

Still, it is beneficial here to study further the ways in which we might expect the varied exception cases of Table III to conspire in their cancellation. One obvious possibility is that a pair of subscribers to the same exception class take opposite assignments of $(Nn\vec{\omega})_a$ and find agreement amongst themselves. However, this is often unrealizable. We note that *between* the cases of **3** *and* **4** any two choices for the vector $\vec{\omega}$ will have exactly *two* or *no* elements in common. If there are two shared basis vectors which have coefficients such to vanish when paired then the remaining sum of two elements will always be such that it can *in principle* be countered by a stack of exception type **1** or *2*. *Within* either case **3** *or* **4** any two $\vec{\omega}$'s have either all three (they are the *same* set of $3\varepsilon_i$) or only one common element. If the single duplicate basis member is canceled then *two* stacks from the set of cases (**1**,**2**) are enough to close the tadpole. However, when dealing here with magnitude 5 tadpole contributions, the quadruple-stack picture *already* over-saturates the bulk. This seems to summarize the reasonable basis scenarios in satisfaction of (4.38).

It is also important to mention some general conditions on the bi-fundamental multiplicity numbers from (4.42,4.43). When taking the intersection between two stacks that contribute each to just a *single* bulk tadpole, there is an *isolated* term that survives if (and only if) the respective stacks hit the opposed tadpoles *and* both

[7]This principle has been well known for some time to carrier-based Naval aircrew.

employ fractional cycles.

$$\pi_a^{(\prime)} \circ \pi_b \sim -\tfrac{1}{2} n_a m_b \vec{\omega}_a \cdot \vec{\omega}_b \tag{4.49}$$

If a single-tadpole stack a intersects a double-type stack b, there are generally two non-vanishing terms, for example:

$$\pi_a^{(\prime)} \circ \pi_b \sim -2 \left\{\tfrac{1}{2}\right\}_a^{\mathrm{f}} \left\{\tfrac{1}{2}\right\}_b^{\mathrm{f}} (n\Delta)_a (m\Delta)_b - \tfrac{1}{2} n_a m_b \vec{\omega}_a \cdot \vec{\omega}_b \tag{4.50}$$

If both stacks are participants in both available bulk tadpole constraints, then all six terms should be expected.

CHAPTER V

CONCLUSION

A. Summary and Conclusions

This study has considered Grand Unified Theories as the essential bridge between low energy phenomena and next generation String Theories. We have reviewed the compelling arguments in favor of the Flipped $SU(5)$ GUT, including simultaneous consistency with $\alpha_s(M_Z)$ and proton lifetime limits. The tremendous expense, both material and intellectual, entailed by the experimental search for proton decay demands a diverse and comprehensive survey of prospects for discovery by the theoretical community. In addition to helping develop consensus on the likelihood that we could be looking in the "right place", laying our cards on the table has the greater benefit of establishing how finely the results, even null results, will discern between competing proposals. The great sensitivity of proton lifetime to the selected GUT structure, and even to the supersymmetric parameter space, do seem to justify the efforts required. We believe the analysis here is the most accurate and comprehensive to date for the flipped picture.

In conjunction, investigations into the two leading paradigms for string model building have been presented. Free Fermionic constructions require assignment of a non-trivial vacuum, oriented in the field space to preserve a flat scalar potential, in order to trim field content, provide masses and cancel the FI anomaly. We have advanced the technology of assigning such VEVs to non-Abelian fields for the Lie groups $SU(2)$ and $SO(2n)$, taking the stance that a geometrical interpretation of the adjoint VEV representation will greatly aid visual intuition of the model builder. A more modern approach employs Intersecting D-Branes for the direct realization of GUT multiplets at the string scale. Focusing on the $T^6/\mathbb{Z}_4$ orientifold, we have undertaken a numerical search for the flipped $SU(5)$ GUT in parallel to an analytic reduction of the simultaneous constraints of supersymmetry and anomaly cancellation on the brane configuration. The rules described allow in some cases a comprehensive manual classification of existing models which include the desired representations.

B. Future Work

I would like to develop an improved understanding of the second stage super-unification which the flipped $SU(5)$ GUT undergoes, presumably at a suitably reduced string scale. Since this energy depends from below on the precision LEP measurements at M_Z and the CMSSM parameter space, besides additional heavy thresholds effects on either side of M_{32}, it is intrinsically linked to low energy particle physics (and also very strongly correlated to proton lifetime). On the other hand, the scale is inherently a string determined value, providing a rare and valuable tangible link to that regime. Looking down the wrong end of the telescope, we may imagine that mechanisms proposed for a slight lightening of the string mass are a necessary consequence of grand unification.

It may also be valuable in time to extend the brief examples studied for application of non-Abelian VEVs, particularly within the existing flipped $SU(5)$ Heterotic construction. Within the D-brane approach, I would like to complete an exhaustive search all $SU(5)$ type theories which exist within $T^6/\mathbb{Z}_4$, developing a better understanding of the role of tilting to influence the spectrum. The proposed "third tilting" also deserves a proper study to determine whether it is truly independent. It would additionally be good to attempt extension of the listed rules, which benefit from purely positive contributions to the anomaly for all stacks, to alternate orientifold configurations. It may be possible to play two (or more) 5-stacks against each other to avoid or cancel the appearance of light symmetric (**15**) representations. Finally, a second generation computer program which benefits from the developed wisdom to greatly reduce required runtime, while maintaining the comprehensive and accurate nature of a numerical search, would be a tremendous development.

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APPENDIX A

Presented here is the choice of $SO(6)$ Matrices in the Original and Diagonal bases [1].

$$\begin{pmatrix} 0 & i & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \overset{M_{12}}{,} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \overset{M_{34}}{,} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & -i & 0 \end{pmatrix} \overset{M_{56}}{,} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix};$$

$$E_1 \begin{bmatrix} + \\ + \\ \times \end{bmatrix} \equiv \frac{1}{2} \begin{pmatrix} 0 & 0 & -1 & i & 0 & 0 \\ 0 & 0 & i & 1 & 0 & 0 \\ 1 & -i & 0 & 0 & 0 & 0 \\ -i & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad E_2 \begin{bmatrix} - \\ - \\ \times \end{bmatrix} \equiv \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & i & 0 & 0 \\ 0 & 0 & i & -1 & 0 & 0 \\ -1 & -i & 0 & 0 & 0 & 0 \\ -i & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

$$E_3 \begin{bmatrix} + \\ - \\ \times \end{bmatrix} \equiv \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & i & 0 & 0 \\ 0 & 0 & -i & 1 & 0 & 0 \\ -1 & i & 0 & 0 & 0 & 0 \\ -i & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad E_4 \begin{bmatrix} - \\ + \\ \times \end{bmatrix} \equiv \frac{1}{2} \begin{pmatrix} 0 & 0 & -1 & i & 0 & 0 \\ 0 & 0 & -i & -1 & 0 & 0 \\ 1 & i & 0 & 0 & 0 & 0 \\ -i & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

$$E_5 \begin{bmatrix} + \\ \times \\ + \end{bmatrix} \equiv \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & -i & -1 \\ 0 & 0 & 0 & 0 & -1 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ i & 1 & 0 & 0 & 0 & 0 \\ 1 & -i & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad E_6 \begin{bmatrix} - \\ \times \\ - \end{bmatrix} \equiv \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & -i & 1 \\ 0 & 0 & 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ i & -1 & 0 & 0 & 0 & 0 \\ -1 & -i & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

$$E_7 \begin{bmatrix} + \\ \times \\ - \end{bmatrix} \equiv \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & i & -1 \\ 0 & 0 & 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -i & -1 & 0 & 0 & 0 & 0 \\ 1 & -i & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \end{pmatrix}; \quad E_8 \begin{bmatrix} - \\ \times \\ + \end{bmatrix} \equiv \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & i & 1 \\ 0 & 0 & 0 & 0 & -1 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 1 & 0 & 0 & 0 & 0 \\ -1 & -i & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

$$E_9 \begin{bmatrix} \times \\ + \\ + \end{bmatrix} \equiv \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & i \\ 0 & 0 & 0 & 0 & i & 1 \\ 0 & 0 & 1 & -i & 0 & 0 \\ 0 & 0 & -i & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad E_{10} \begin{bmatrix} \times \\ - \\ - \end{bmatrix} \equiv \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 & i & -1 \\ 0 & 0 & -1 & -i & 0 & 0 \\ 0 & 0 & -i & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix};$$

$$E_{11} \begin{bmatrix} \times \\ + \\ - \end{bmatrix} \equiv \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -i \\ 0 & 0 & 0 & 0 & i & -1 \\ 0 & 0 & 1 & -i & 0 & 0 \\ 0 & 0 & i & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}; \quad E_{12} \begin{bmatrix} \times \\ - \\ + \end{bmatrix} \equiv \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -i \\ 0 & 0 & 0 & 0 & i & 1 \\ 0 & 0 & -1 & -i & 0 & 0 \\ 0 & 0 & i & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

[1] The symbols $(+, -, \times)$ describe the raising, lowering or null role of the operators E_i in each of the three selected diagonal sectors.

APPENDIX B

Presented following are programs for numerical extraction of gauged models on $T^6/\boldsymbol{Z}_4$.

1. Phase 1: The Supersymmetry Condition

```
#!/usr/bin/perl

#T6/Z4

require ’ibw_subrouts.p’;

use vars qw(@cop $maxindex);

my @inv = (’AAA’, ’ABA’, ’AAB’, ’ABB’);
my $tori = 3;

my ($n1,$m1,$n2,$m2,$n3,$m3,$wrap,$gamma,$delta,$u2,$cs);

$maxindex = 1;

&COPRIME;

&WRAPPINGNUMBERS;

&SORT;

sub WRAPPINGNUMBERS {
   my (@index,$cp);
   my $t = $tori - 1;
   my $cop = $#cop;
   for (@inv) {
      open (IBW, ’>SUSY/IBW_SUSY_’.$_.’_’.$maxindex);
      my ($i);
      my @cs = split ’’;
      *NM2 = *{’NM_2’.$cs[1]};
      *NM3 = *{’NM_3’.$cs[2]};
      ($index[$_] = $cop) for (0..$t);
      until ($i < 0) {
         $wrap = join ’ ’, (($n1,$m1,$n2,$m2,$n3,$m3) = map @{$cop[$_]}, @index);
         ($n2,$m2) = &NM2;
         ($n3,$m3) = &NM3;
         ($gamma,$delta) = &TAN3;
         for (&COMPLEXSTRUCTURE) {
            ($u2,$cs) = @$_;
            next unless $cs;
            $m3 *= $u2;
            if (&QUADRANT) {
               print IBW $cs, "\t"x(length($cs)>7?1:2), $wrap, "\n"; }}}
      continue {
         $i = $t;
```

```
         ($index[$i--] = $cop) while (--$index[$i] < 0); }
      close IBW; }}

sub SORT {
   for (map $_, @inv) {
      my %cs;
      my $file = 'SUSY/IBW_SUSY_'.$_.'_'.$maxindex;
      open (IBW, $file);
      while (<IBW>) {
         my ($k,$v) = split /\s+/, $_, 2;
         push @{$cs{$k}}, $v; }
      close IBW;
      open (IBW, '>'.$file);
      select IBW;
      for (sort {$b cmp $a} keys %cs) {
         print $_, "\t", int @{$cs{$_}}, "\n\t";
         print join "\t", @{$cs{$_}}; }
      close IBW; }}

sub COMPLEXSTRUCTURE {
   if ($gamma == 0) {
      ($m3 == 0) ? [0,'ANY'] : [0,0]; }
   elsif ($delta == 0) {
      ($n3 == 0) ? ([1,'POS'],[-1,'NEG']) : [0,0]; }
   else {
      return [0,0] if ($n3*$m3 == 0);
      my $cs = 1.0*($n3/$m3*$gamma/$delta);
      [$cs,&ROUNDFIVE($cs)]; }}

sub QUADRANT {
   if (($n1*$m1*$n2*$m2 < 0) or ($gamma == 0)) {
      ($n3*$delta < 0); }
   else {
      ($m3*$gamma < 0); }}

sub NM_2A {
   ($n2,$m2); }

sub NM_2B {
   (($n2+$m2),($m2-$n2)); }

sub NM_3A {
   ($n3,$m3); }

sub NM_3B {
   (($n3+$m3/2),$m3); }

sub TAN3 {
   (($n1*$m2+$m1*$n2),($m1*$m2-$n1*$n2)); }
```

2. Phase 2: The Bulk Cycles

```
#!/usr/bin/perl

#T6/Z4

$| = 1;

require 'ibw_subrouts.p';

my (@bet,@bem,@cs,@rho,@prime,@tad,%any,@rhoprime,$o6);

my @inv = ('AAA', 'ABA', 'AAB', 'ABB');
my @stacks = (5,5,1);
@stacks = reverse sort {$a<=>$b} @stacks;
my $stacks = join '', @stacks;
my $s = $#stacks;
my $is = int @stacks;
my @choose = map { my @tmp = @$_;
   [ map { ($tmp[$_] == 1) ? $_ : () } (0..$s) ]; } &CHOOSENOFM(2,$is);
my $tori = 3;
my $ele = 2*$tori;
my $maxindex = 5;
my @BE = ('B', 'E');
my %BE = ( 'B', 1, 'E', .5 );

my %RHO_PRIME = (
   'AAA' => [
   [ 1, 0, 0, 0],
   [ 0,-1, 0, 0],
   [ 0, 0,-1, 0],
   [ 0, 0, 0, 1] ],
   'ABA' => [
   [ 0, 1, 0, 0],
   [ 1, 0, 0, 0],
   [ 0, 0, 0,-1],
   [ 0, 0,-1, 0] ],
   'AAB' => [
   [ 1, 0, 1, 0],
   [ 0,-1, 0,-1],
   [ 0, 0,-1, 0],
   [ 0, 0, 0, 1] ],
   'ABB' => [
   [ 0, 1, 0, 1],
   [ 1, 0, 1, 0],
   [ 0, 0, 0,-1],
   [ 0, 0,-1, 0] ] );

my %O_SIX = (
   'AAA' => [ 4, 0, 0,-2],
   'ABA' => [ 2, 2, 2,-2],
   'AAB' => [ 2, 1, 0,-2],
   'ABB' => [ 0, 2, 2,-2] );

for (0..$is) {
   push @bet, (map { join '', (map $BE[$_], @$_) } &CHOOSENOFM($_,$is)); }
for (@bet) {
   my @tmp = split //;
   push @bem, [ map { $stacks[$_]*$BE{$tmp[$_]}} (0..$s) ]; }

&BULK;

sub BULK {
   my (@index,@wrap,$wrap,@wrapc,@indexb,@bstart,@wrapb,$wrapb);
```

```
   my $s = $#stacks;
   for (map $_, @inv) {
      my ($cs);
      @rhoprime = @{$RHO_PRIME{$_}};
      $o6 = $O_SIX{$_};
      open (IBW, 'SUSY/IBW_SUSY_'.$_.'_'.$maxindex);
      open (BLK, '>BULK/IBW_BLK_'.$_.'_'.$stacks.'_'.$maxindex);
      select BLK;
      $| = 1;

      while (<IBW>) {
         chop;
         if (s/^\s+//) {
            push @cs, $_; }
         else {
            &STACKS($cs);
            $cs = (split /\s+/, $_, 2)[0];
            @cs = (); }}
      &STACKS($cs);

      close BLK;
      close IBW; }}

sub STACKS {
   my (@index,$i,@be,@wrap,$wrap);
   my $cs = $_[0];
   return unless length $cs;
   @cs = &SYMMETRY(@cs);
   if ($cs =~ /^(ANY|POS|NEG)$/) {
      $any{$cs} = [ @cs ];
      return; }
   my $craw = $#cs;
   push @cs, @{$any{'ANY'}};
   push @cs, @{$any{($cs>0 ? 'POS' : 'NEG')}};
   my $c = $#cs;
   @rho = map { &RHO($_) } @cs;
   @prime = map { &PIPRIME($_) } @rho;
   @tad = map { &VSUM($rho[$_],$prime[$_]) } (0..$c);
   ($index[$_] = $c) for (0..$s);
   until ($i < 0) {
      $i = $s;
      @wrap = map $tad[$_], @index;
      next unless (@be = map { &TADPOLE(@{$bem[$_]},-4,@wrap,$o6) ?
         $bet[$_] : () } (0..$#bem));
      print +(int(map { ($_>$craw) and () } @index) ?
         ($cs.(length($cs)>7?'':"\t")) : "?\t"), "\t";
      print +(join "\t", map $cs[$_], @index), "\t";
      print +(join "\t", &MULTIPLICITIES(@index)), "\t";
      print +(join ' ', @be), "\n"; }
   continue {
      ($index[$i--] = $c) while (--$index[$i] < 0); }}

sub SYMMETRY {
   my @cs = @_;
   my ($i, $j, $k, $sym, @sym, %sub, @sub);
   for $i (0..2) {
      my (%sym);
      J: for $j (0..$#cs) {
         next if $sym{$j};
         $sym = &SYMTRANS($i,$cs[$j]);
         for $k (0..$#cs) {
            next if $sym{$k};
            if ($cs[$k] eq $sym) {
```

```
                push @{$sym[$j]}, $k;
                push @{$sym[$k]}, $j;
                $sym{$j} = $sym{$k} = 1;
                next J; }}}}
   for (0..$#cs) {
      next if $sub{$_};
      push @sub, $_;
      my ($i,$j);
      my %sym = ($_,1);
      until (($j = int keys %sym) == $i) {
         $i = $j;
         %sym = map {($_,1)} map {($_, @{$sym[$_]})} keys %sym; }
      $sub{$_} = 1 for keys %sym; }
   @cs[@sub]; }

sub SYMTRANS {
   my @w = split ’ ’, $_[1];
   if ($_[0] == 0) {
      @w = (-$w[0],-$w[1],-$w[2],-$w[3],$w[4],$w[5]); }
   elsif ($_[0] == 1) {
      @w = ($w[1],-$w[0],-$w[3],$w[2],$w[4],$w[5]); }
   else {
      @w = ($w[0],$w[1],-$w[2],-$w[3],-$w[4],-$w[5]); }
   @w = map $_*1, @w;
   return join ’ ’, @w; }

sub RHO {
   my ($n1,$m1,$n2,$m2,$n3,$m3) = split /\s+/, $_[0];
   my $gam = ($n1*$n2-$m1*$m2);
   my $del = ($n1*$m2+$m1*$n2);
   [$gam*$n3,$del*$n3,$gam*$m3,$del*$m3]; }

sub PIPRIME {
   my (@piprime);
   my @rho = @{$_[0]};
   return ’NULL’ unless (@rho == 4);
   $piprime[$_] = &DOT(@{$rhoprime[$_]},@rho) for (0..3);
   \@piprime; }

sub TADPOLE {
   my $v = @_/2;
   return ’NULL’ unless &ISINTEGER($v);
   for $i (0..3) {
      my @exc = map ${$_[$_]}[$i], ($v..$#_);
      return 0 unless (&DOT(@_[0..$v-1],@exc) == 0); }
   1; }

sub MULTIPLICITIES {
   my (@intx,$intx);
   my @index = @_;
   for (@index) {
      push @intx, join ’ ’, (.5*&CIRCLE(@{$prime[$_]},@{$rho[$_]}),
         .5*&CIRCLE(@{$o6},@{$rho[$_]})); }
   for $tmp (&CHOOSENOFM(2,int @stacks)) {
      my ($a,$b) = @index[(map { ($$tmp[$_] == 1) ? $_ : () } (0..$s))];
      push @intx, (length($intx)<8 ? "\t" : ’’) . ($intx = join ’ ’,
         (&CIRCLE(@{$rho[$a]},@{$rho[$b]}),&CIRCLE(@{$prime[$a]},@{$rho[$b]}))); }
   @intx; }

sub VSUM {
   my ($i, @vsum);
   for $i (0..$#{$_[0]}) {
      $vsum[$i] += $$_[$i] for @_; }
```

```
   \@vsum; }

sub DOT {
   my ($dot);
   my $v = @_/2;
   return 'NULL' unless &ISINTEGER($v);
   $dot += $_[$_]*$_[$_+$v] for (0..$v-1);
   $dot; }

sub CIRCLE {
   return 'NULL' unless (@_ == 8);
   (2*&DOT(@_[2,3],@_[4,5]) - 2*&DOT(@_[0,1],@_[6,7])); }
```

3. Phase 3: The Exceptional Cycles

```
#!/usr/bin/perl

#T6/Z4

$| = 1;

require 'ibw_subrouts.p';

my @inv = ('AAA', 'ABA', 'AAB', 'ABB');
my @stacks = (5,5,1);
@stacks = reverse sort {$a<=>$b} @stacks;
my $stacks = join '', @stacks;
my $s = $#stacks;
my $d = ($s*($s+1)/2 - 1);
my @choose = map { my @tmp = @$_;
   [ map { ($tmp[$_] == 1) ? $_ : () } (0..$s) ]; } &CHOOSENOFM(2,int @stacks);
my $tori = 3;
my $ele = 2*$tori;
my $maxindex = 5;
my %BE = ( 'B', 1, 'E', .5 );

my ($cindx,@cycles,@excindx,@tadpole,@mult,@AAA_PRIME,@ABA_PRIME,@AAB_PRIME,@ABB_PRIME);
my ($prime);

my @exc = (
   [[1,1,0,0,0,0], [0,0,0,0,1,1]],
   [[0,0,1,1,0,0], [0,0,0,0,1,1]],
   [[1,0,1,0,1,0], [1,0,0,1,0,1], [0,1,1,0,0,1], [0,1,0,1,1,0]],
   [[1,0,1,0,0,1], [1,0,0,1,1,0], [0,1,1,0,1,0], [0,1,0,1,0,1]],
   [[0,0,0,0,0,0]] );

my @blank = ( 0,0,0,0,0,0 );

&PRIME;

&CYCLES;

&EXCEPTIONS;

&SPECT;

sub EXCEPTIONS {
   my (@be,$cs,@wrap,@self,@dual,@ei,$wrap,@si,@di,@mi,@index,@wrap3,@wrapE);
   for (map $_, @inv) {
      open (BLK, 'BULK/IBW_BLK_'.$_.'_'.$stacks.'_'.$maxindex);
      open (IBW, '>EXCEPTIONS/IBW_EXC_'.$_.'_'.$stacks.'_'.$maxindex);
      select IBW;
      $| = 1;
      $prime = \@{$_.'_PRIME'};

      &INTERSECTIONS;

      while (<BLK>) {
         chop;
         @be = split /\s+/;
         $cs = shift @be;
         @wrap = map [ splice (@be,0,$ele) ], @stacks;
         @self = map [ splice (@be,0,2) ], @stacks;
         @dual = map [ splice (@be,0,2) ], (0..$d);
         @ei = map $excindx[&EXCINDX(@$_)], @wrap;
         $wrap = '(' . (join ') (', (map { join '', @$_ } @wrap)) . ')';
```

```
         $cs .= "\t" x ((length($cs)<8) ? 2 : 1);

         BE: for (@be) {
            my ($i,@si,@di);
            my @bet = split //, $_;
            my @bei = @ei;
            for (0..$s) {
               if ($bet[$_] eq 'E') {
                  next BE if (($mi[$_] = $#{$bei[$_]}) == -1);
                  my ($a,$b) = @{$self[$_]};
                  push @si, [.25*$a, .5*$b]; }
               else {
                  $mi[$_] = 0;
                  $bei[$_] = [$cindx];
                  push @si, $self[$_]; }}
            @index = @mi;

            @si = map [ ($$_[0]+$$_[1], $$_[0]-$$_[1]) ], @si;
            @di = map { my ($a,$b) = @BE{@bet[@{$choose[$_]}]};
               my $c = $a*$b;
               [ map $c*$_, @{$dual[$_]} ]; } (0..$d);

            until ($i < 0) {
               my ($ints,$intd);
               @wrap3 = map { ($stacks[$_]*${$wrap[$_]}[4],
                  $stacks[$_]*${$wrap[$_]}[5]) } (0..$s);
               @wrapE = map ${$bei[$_]}[$index[$_]], (0..$s);

               next unless &TADPOLE(@wrap3,
                  (map @{$tadpole[$_]}, @wrapE));
               for (0..$s) {
                  my ($a,$n,$m) = ($wrapE[$_],@{$wrap[$_]}[4,5]);
                  my @coef = ($n*$n,$m*$m,2*$n*$m);
                  my $si = .5*&DOT(@coef,@{$mult[$a][$a]}[0..2]);
                  $ints = join ' ', (map $si+$_, @{$si[$_]});
                  print $ints, "\t" x ((length($ints)<8) ? 2 : 1); }
               print '   ||   ';
               my @dit = @di;
               for (@choose) {
                  my ($g,$n1,$m1,$d,$n2,$m2) =
                     map { ($wrapE[$_],@{$wrap[$_]}[4,5]) } @$_;
                  ($g,$d) = reverse sort {$a<=>$b} ($g,$d);
                  my @d = @{$mult[$g][$d]};
                  my @coef = ($n1*$n2,$m1*$m2,($n1*$m2+$m1*$n2));
                  $intd = join ' ', @{ &VSUM(shift @dit,
                     [($n1*$m2-$m1*$n2)*(pop @d),&DOT(@coef,@d)]) };
                  print $intd, "\t" x ((length($intd)<8) ? 2 : 1); }
               print '   ||   ';
               @wrapE = map {($_ == $cindx) ? 'B' : $_} @wrapE;
               my $wrapE = '(' . (join ',', @wrapE) . ')';
               print $cs, $wrapE,
                  "\t" x ((length($wrapE)<8) ? 2 : 1), $wrap, "\n"; }
            continue {
               $i = $s;
               ($index[$i--] = $mi[$i]) while (--$index[$i] < 0); }}}
      close IBW;
      close BLK; }}

sub CYCLES {
   open (IBW, '>EXCEPTIONS/IBW_CYCLES');
   print IBW '-'x25, "\n";
   for (@exc) {
      for $tmp (@$_) {
```

```
      push @cycles, $tmp;
      my @active = map { ($$tmp[$_] == 1) ? $_ : () } (0..5);
      my $act = @active;
      for $n (1..$act) {
         for (&CHOOSENOFM($n,$act)) {
            my @cycle = @blank;
            @cycle[@active] = map { (-2*$_) + 1 } @$_;
            push @cycles, \@cycle; }}}
   push @excindx, [($cindx..$#cycles)];
   (print IBW $_, ":\t", (join ' ', @{$cycles[$_]}), "\n") for @{$excindx[-1]};
   print IBW '-'x25, "\n";
   $cindx = @cycles; }
 close IBW;
 $cindx--; }

sub SPECT {
   for (map $_, @inv) {
      my (%spect);
      my $file = 'EXCEPTIONS/IBW_EXC_'.$_.'_'.$stacks.'_'.$maxindex;
      open (IBW, $file);
      while (<IBW>) {
         chop;
         /^(.*?\|\|.*?)\s+\|\|\s+(.*)$/;
         push @{$spect{$1}}, $2; }
      close IBW;
      open (IBW, '>'.$file);
      select IBW;
      print +(join "\t\t", @stacks), "\t\t", '   ||   ';
      print +(join "\t\t", map {$stacks[$$_[0]].
         '_'.$stacks[$$_[1]]} @choose), "\n";
      print "\t", 'COMPLEX STRUCT', "\t", '(EXC INDEX)',
         "\t", '(WRAPPING NUMBERS)', "\n";
      print +('-' x (16*($s+$d+2))), "\n";
      for (keys %spect) {
         print $_, "\n";
         (print "\t", $_, "\n") for @{$spect{$_}}; }
      close IBW; }}

sub EXCINDX {
   my ($n1,$m1,$n2,$m2,@nm3) = @_;
   my ($a,$b) = (($n1+$n2),($n1*$m1));
   if (&ISEVEN($a+$m1+$m2)) {
      return (&ISEVEN($b) ? 2 : 4) if (&ISEVEN($a));
      return 3; }
   else {
      return (&ISEVEN($b) ? 1 : 0); }}

sub INTERSECTIONS {
   my @un = map { [@$_, @blank] } @cycles;
   my @bar = map { [@blank, @$_] } @cycles;
   my @unp = map { [&PIPRIME(@$_)] } @un;
   my @barp = map { [&PIPRIME(@$_)] } @bar;

   for $i (0..$#cycles) {
      $tadpole[$i] = [&VSUM($un[$i],$unp[$i]),
         &VSUM($bar[$i],$barp[$i])];

      for $j (0..$i) {
         $mult[$i][$j] = [
            .25*&CIRCLE(@{$unp[$i]},@{$un[$j]}),
            .25*&CIRCLE(@{$barp[$i]},@{$bar[$j]}),
            .25*&CIRCLE(@{$unp[$i]},@{$bar[$j]}),
            .25*&CIRCLE(@{$un[$i]},@{$bar[$j]}) ]; }}}
```

```
sub TADPOLE {
   my $v = @_/2;
   return 'NULL' unless &ISINTEGER($v);
   for $i (0..11) {
      my @exc = map ${$_[$_]}[$i], ($v..$#_);
      return 0 unless (&DOT(@_[0..$v-1],@exc) == 0); }
   1; }

sub ISEVEN {
   &ISINTEGER($_[0]/2); }

sub VSUM {
   my ($i, @vsum);
   for $i (0..$#{$_[0]}) {
      $vsum[$i] += $$_[$i] for @_; }
   \@vsum; }

sub DOT {
   my ($dot);
   my $v = @_/2;
   return 'NULL' unless &ISINTEGER($v);
   $dot += $_[$_]*$_[$_+$v] for (0..$v-1);
   $dot; }

sub CIRCLE {
   return 'NULL' unless (@_ == 24);
   (2*&DOT(@_[6..11],@_[12..17]) - 2*&DOT(@_[0..5],@_[18..23])); }

sub PIPRIME {
   my (@piprime);
   return 'NULL' unless (@_ == 12);
   for (0..11) {
      $piprime[$_] = &DOT(@{${$prime}[$_]},@_); }
   @piprime; }

sub PRIME {
   @{'AAA_PRIME'} = (
      [ 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
      [ 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
      [ 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0],
      [ 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0],
      [ 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0],
      [ 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0],
      [ 0, 0, 0, 0, 0, 0,-1, 0, 0, 0, 0, 0],
      [ 0, 0, 0, 0, 0, 0, 0,-1, 0, 0, 0, 0],
      [ 0, 0, 0, 0, 0, 0, 0, 0,-1, 0, 0, 0],
      [ 0, 0, 0, 0, 0, 0, 0, 0, 0,-1, 0, 0],
      [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,-1, 0],
      [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,-1] );
   @{'ABA_PRIME'} = (
      [-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
      [ 0,-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
      [ 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0],
      [ 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0],
      [ 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0],
      [ 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0],
      [ 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0],
      [ 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0],
      [ 0, 0, 0, 0, 0, 0, 0, 0,-1, 0, 0, 0],
      [ 0, 0, 0, 0, 0, 0, 0, 0, 0,-1, 0, 0],
      [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,-1],
      [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,-1, 0] );
   @{'AAB_PRIME'} = (
```

```
    [ 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0],
    [ 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0],
    [ 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0],
    [ 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0],
    [ 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0],
    [ 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1],
    [ 0, 0, 0, 0, 0, 0,-1, 0, 0, 0, 0, 0],
    [ 0, 0, 0, 0, 0, 0, 0,-1, 0, 0, 0, 0],
    [ 0, 0, 0, 0, 0, 0, 0, 0,-1, 0, 0, 0],
    [ 0, 0, 0, 0, 0, 0, 0, 0, 0,-1, 0, 0],
    [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,-1, 0],
    [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,-1] );
  @{'ABB_PRIME'} = (
    [-1, 0, 0, 0, 0, 0,-1, 0, 0, 0, 0, 0],
    [ 0,-1, 0, 0, 0, 0, 0,-1, 0, 0, 0, 0],
    [ 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0],
    [ 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0],
    [ 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1],
    [ 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0],
    [ 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0],
    [ 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0],
    [ 0, 0, 0, 0, 0, 0, 0, 0,-1, 0, 0, 0],
    [ 0, 0, 0, 0, 0, 0, 0, 0, 0,-1, 0, 0],
    [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,-1],
    [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,-1, 0] ); }
```

4. Subroutines

```
#!/usr/bin/perl

#T6/Z4

sub COPRIME {
   my ($a, $b);
   my @div = map {
      my $a = $_;
      my %div = map {&ISINTEGER($a/$_) ? ($_,1) : ()} (2..$a/2);
      \%div; } (0..$maxindex);
   @cop = ();
   A: for $a (2..$maxindex) {
   B: for $b (1..$a-1) {
      next B if (map {${$div[$a]}{$_} or ()} (keys %{$div[$b]},$b));
      push @cop, [$b,$a]; }}
   push @cop, map {
      ($a,$b) = @$_;
      ([-$a,$b],[$a,-$b],[-$a,-$b],[$b,$a],[-$b,$a],[$b,-$a],[-$b,-$a]); } @cop;
   push @cop, ([0,1],[0,-1],[1,0],[-1,0],[1,1],[-1,1],[1,-1],[-1,-1]); }

sub CHOOSENOFM {
   my ($n, $m) = @_;
   my (@perm);
   $perm[0] = [ map 0, 1..$m ];
   for $i (1..$n) {
      my (@nofm);
      for (@perm) {
         my @set = @$_;
         my $zero = $m;
         for (reverse @set) {
            last if ($_ == 1);
            $zero--; }
         for ($zero..($m-$n+$i-1)) {
            my @iofm = @set;
            $iofm[$_]++;
            push @nofm, \@iofm; }}
      @perm = @nofm; }
   return @perm; }

sub ISINTEGER {
   $_[0] == int $_[0]; }

sub ROUNDFIVE {
   .00001 * int (100000*$_[0] + ($_[0]>0?.5:-.5)); }

1;
```

VITA

Joel W. Walker received his Bachelor of Science degree in physics from Harding University in Searcy, Arkansas in 1997. He entered graduate study in physics at Texas A&M University in the fall of that year, and received his Doctor of Philosophy degree in August 2005. Under the supervision of Dr. Dimitri Nanopoulos, his research and study interests have included particle physics, cosmology, grand unification, and superstring phenomenology. Computer programming and numerically assisted parameter space reduction have been a consistent element of his research program.

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